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# NAVAL POSTGRADUATE SCHOOL

## Monterey, California



## THESIS

### AXIAL CONDUCTION EFFECTS IN LAMINAR DUCT FLOWS

by

Ibrahim Girgin

June, 1998

Thesis Advisor:

Ashok Gopinath

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**AXIAL HEAT CONDUCTION EFFECTS IN  
LAMINAR DUCT FLOWS**

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Submitted in partial fulfillment of the  
requirements for the degree of

**MASTER OF SCIENCE IN MECHANICAL ENGINEERING**

from the

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## ABSTRACT

A numerical model for heat transfer in laminar duct flows has been developed using the finite difference method to explore the significance and extent of “back-conduction” at low Peclet numbers. The calculations have been carried out for flows between parallel plates and in circular tubes by using different Peclet numbers in the range of 0.05 to 100. For both situations constant heat flux and constant wall temperature boundary conditions were used. The validity of the results has been checked by comparison with some existing results in the literature, and extended to a wider range of parameters including conjugate wall conduction effects. The results are presented for bulk mean temperature variation, Nusselt number behavior, and energy absorbed before the heated section, for cases with and without wall conduction. Such axial conduction effects may be an important feature in the thermal characterization of microtubes, which are to be used in microheat exchangers.



# TABLE OF CONTENTS

I. INTRODUCTION.....	1
II. BACKGROUND.....	3
A. HEAT TRANSFER IN LAMINAR DUCT FLOWS.....	3
1. The Heat Transfer.....	5
2. The Thermally Fully Developed Flow.....	6
B. BACKGROUND STUDIES.....	6
C. NUMERICAL METHOD.....	8
III. GOVERNING EQUATIONS.....	11
A. DERIVATION OF THE EQUATIONS.....	11
1. Flow Inside the Circular Tube, Constant Wall Temperature Case.....	11
2. Flow Inside the Circular Tube, Constant Heat Flux Case.....	12
3. Flow Between the Parallel Plates, Constant Wall Temperature Case.....	12
4. Flow Between the Parallel Plates, Constant Heat Flux Case.....	14
5. Effects of Wall Conduction, Flow Inside the Circular Tube, Constant Wall Temperature.....	14
B. BOUNDARY CONDITIONS.....	15
1. Flow Inside the Circular Tube, Constant Wall Temperature Case .....	15
2. Flow Inside the Circular Tube, Constant Heat Flux Case.....	16
3. Flow Between the Parallel Plates, Constant Wall Temperature Case.....	17
4. Flow Between the Parallel Plates, Constant Heat Flux Case.....	18
5. Effects of Wall Conduction, Flow Inside the Circular Tube, Constant Wall Temperature.....	19
IV. NUMERICAL REPRESENTATION.....	21
A. FLOW INSIDE THE CIRCULAR TUBE, CONSTANT WALL TEMPERATURE CASE.....	21
1. The Governing Equation.....	21
2. The Boundary Conditions.....	22
3. The Bulk Mean Temperature.....	24
4. The Nusselt Number.....	25
5. Heat Absorption Upto the Entrance.....	26
B. FLOW INSIDE THE CIRCULAR TUBE, CONSTANT HEAT FLUX CASE.....	27
1. The Governing Equation.....	27
2. The Boundary Conditions.....	28
3. The Bulk Mean Temperature.....	31
4. The Nusselt Number.....	32
5. Heat Absorption Upto the Entrance.....	32



C. FLOW BETWEEN THE PARALLEL PLATES, CONSTANT WALL TEMPERATURE CASE.....	33
1. The Governing Equation.....	33
2. The Boundary Conditions.....	34
3. The Bulk Mean Temperature.....	36
4. The Nusselt Number.....	37
5. Heat Absorption Upto the Entrance.....	38
D. FLOW BETWEEN THE PARALLEL PLATES, CONSTANT HEAT FLUX CASE.....	39
1. The Governing Equation.....	39
2. The Boundary Conditions.....	40
3. The Bulk Mean Temperature.....	42
4. The Nusselt Number.....	43
5. Heat Absorption Upto the Entrance.....	44
E. EFFECTS OF WALL CONDUCTION, FLOW INSIDE THE CIRCULAR TUBE, CONSTANT WALL TEMPERATURE CASE .....	45
1. The Governing Equation.....	45
2. The Boundary Conditions.....	46
3. The Bulk Mean Temperature.....	49
4. The Nusselt Number.....	50
5. Heat Absorption Upto the Entrance.....	50
V. NUMERICAL RESULTS AND DISCUSSION.....	53
A. BULK MEAN TEMPERATURE VS. X.....	53
1. Flow Inside the Circular Tube, Constant Wall Temperature Case.....	53
2. Flow Inside the Circular Tube, Constant Heat Flux Case.....	55
3. Flow Between the Parallel Plates, Constant Wall Temperature Case.....	56
4. Flow Between the Parallel Plates, Constant Heat Flux Case.....	57
B. NUSSULT NUMBER VS. X.....	58
1. Flow Inside the Circular Tube, Constant Wall Temperature Case.....	58
2. Flow Inside the Circular Tube, Constant Heat Flux Case.....	59
3. Flow Between the Parallel Plates, Constant Wall Temperature Case.....	60
4. Flow Between the Parallel Plates, Constant Heat Flux Case.....	61
C. HEAT ABSORPTION UPTO THE ENTRANCE.....	61
1. Flow Inside the Circular Tube, Constant Wall Temperature Case.....	62
2. Flow Inside the Circular Tube, Constant Heat Flux Case.....	62
3. Flow Between the Parallel Plates, Constant Wall Temperature Case.....	63
4. Flow Between the Parallel Plates, Constant Heat Flux Case.....	64
D. AXIAL CONDUCTION EFFECTS INCLUDING THE WALL CONDUCTION	64
1. Bulk Mean Temperature Variations.....	65
2. Nusselt Number vs. X.....	67
3. Heat Absorption vs. X.....	68
VI. CONCLUSIONS.....	69

APPENDIX – A SAMPLE PROGRAM .....71

LIST OF REFERENCES.....75

INITIAL DISTRIBUTION LIST.....77





## LIST OF SYMBOLS

SYMBOL	DESCRIPTION	UNITS
$c_p$	Specific heat at constant pressure	J/kg.K
$d$	The diameter of the circular tube	m
$D_h$	Hydraulic diameter for parallel plates, $D_h=4h$	m
$E_{\text{absorbed}}$	Absorbed energy inside the fluid from the entrance to $x=0$	Joule
$E^*$	Dimensionless absorbed energy	unity
$h$	Convection heat transfer coefficient	W/m <sup>2</sup> .K
$h$	The distance from the centerline to the surface of the parallel plate	m
$h_x$	The distance between two grid points in x direction	m
$h_y$	The distance between two grid points in y direction	m
$k$	Thermal conductivity	W/m.K
$Nu$	Nusselt number	unity
$Pe, Pe_{Dh}$	Peclet number, $Pe=Re.Pr$	unity
$Pr$	Prandtl number	unity
$q_o^*$	Heat flux from the surface in constant heat flux case	W/m <sup>2</sup>
$r$	Distance from the center of the circular tube	m
$r_o$	Radius of the circular tube	m
$r^+$	Dimensionless radius, $r/r_o$	unity
$Re$	Reynolds number	unity
$T, t$	Temperature	°K, °C
$t_w, t_o$	Wall temperature	°K, °C
$t_e$	Fluid entrance temperature	°K, °C
$T_m, t_m$	Mean temperature	°K, °C
$u$	Velocity	m/sec
$u^+$	Dimensionless velocity	unity
$V$	Average velocity	m/sec
$v_r$	Radial velocity of the fluid inside the circular tube	m/sec
$w$	Relaxation parameter	unity
$x^+$	Dimensionless distance in x direction	unity
$y$	Distance from the centerline of the parallel plates	m
$\nu$	Kinematic viscosity	m <sup>2</sup> /sec
$\alpha$	Thermal diffusivity	m <sup>2</sup> /sec
$\mu$	Viscosity	kg/sec.m
$\rho$	Density	kg/m <sup>3</sup>
$\theta$	Bulk temperature	unity
$\theta_m$	Bulk mean temperature	unity



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I would like to dedicate this thesis to all of my loved ones and my family. A very special thanks to my lovely wife *Funda* who has been an inspiration for me even though being on the other side of the earth. Thanks to my dear friends who have always supported me in every cases.





# I. INTRODUCTION

The heat transfer problem of laminar fluid flow in ducts which is known as the Graetz problem has many applications in technology and has been studied extensively since Graetz (1885). The classical Graetz problem considers the forced convection heat transfer of the fluids flowing in ducts neglecting axial conduction effects in the fluid or the wall.

This study is on the axial conduction effects of the flows whose Peclet numbers range is from those of liquid metals where axial conduction may not be neglected, to those of oils where the axial conduction has almost no effect on the temperature distribution inside the ducts. Developing micromachines nowadays and in the future, the axial conduction effects may be a very important feature in microheat exchangers.

The effects of axial conduction on the Hagen-Poiseuille flows between parallel plates and in circular cylinders for constant wall temperature and constant heat flux have been studied in this thesis. The governing equations have been solved using the finite difference numerical scheme. The validity of the results has been checked by comparison with some existing results in the literature and extended to a wider range of parameters including conjugate wall conduction effects. The results are presented for bulk mean temperature variation, Nusselt number behavior, and energy absorbed before the heated section, for cases with and without wall conduction.



## II. BACKGROUND

### A. HEAT TRANSFER IN LAMINAR DUCT FLOWS

Consider a flow in a circular tube where fluid enters the tube with a uniform velocity. The viscous effects are important in this flow and a boundary layer develops as  $x$  increases. The flow is “fully developed” in the region where the velocity gradient doesn’t change anymore with increasing  $x$  (Figure 1).

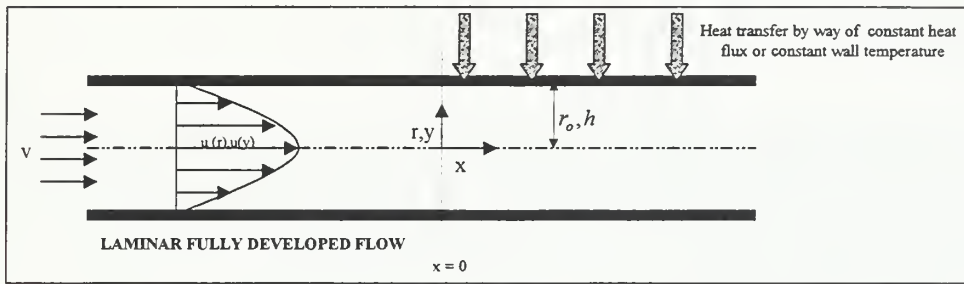


Figure 1. Laminar fully developed flow

The Reynolds Number for the duct flows is defined as

$$Re_D = \frac{VD_H}{\nu} \quad (2.1)$$

where  $V$  is the mean fluid velocity,  $\nu$  is kinematic viscosity,  $\nu = \frac{\mu}{\rho}$ , and  $D_H$  is the hydraulic diameter,  $D_H = 4 \frac{\text{Cross\_Sectional\_Area}}{\text{Wetted\_Perimeter}}$ . For the laminar duct flows,  $Re_D \leq 2300$ .

The Prandtl number relates the temperature distribution to the velocity distribution and defined as

$$Pr = \frac{\nu}{\alpha} \quad (2.2)$$

where  $\alpha$  is the thermal diffusivity . If the Prandtl number is one, the velocity and the temperature profiles develop together and at the same rate. The Prandtl number is:

$Pr \ll 1$  for the liquid metals

$Pr \cong 1$  for the gases

$Pr \gg 1$  for the oils

For the liquid metals, where the  $Pr \ll 1$ , the energy diffusion rate is much more than the momentum diffusion rate. It is opposite for the oils, in which  $Pr \gg 1$  and the velocity profile develops faster than the temperature profile in this case. The value of the Prandtl number strongly affects the relative growth of the velocity and thermal boundary layers.

Peclet number is defined as

$$Pe = Re \cdot Pr \quad (2.3)$$

and it is a measure of the quantity of the axial heat conduction effects in the fluid. The axial conduction is assumed negligible for  $Pe > 10$  and the axial conduction term can be assumed small in the governing equation for this case. But the axial conduction effects can be significant when the Peclet number is smaller. The purpose of this numerical study is to show the importance of the axial conduction effects and the heat absorption upto the entrance from where the heating starts when the Peclet number is small.

The Nusselt number is defined as

$$Nu = \frac{hD_h}{k} \quad (2.4)$$

where  $k$  is the thermal conductivity of the fluid. It is a non-dimensional number where the convection coefficient  $h$  is calculated.

The laminar duct flow being considered is assumed to be hydrodynamically fully developed before any heating effects are considered. The velocity profile in the laminar fully developed region is:

For the circular tube:

$$u(r) = 2V \left[ 1 - \left( \frac{r}{r_o} \right)^2 \right] \quad (2.5)$$



For the parallel plates:

$$u(y) = \frac{3}{2} V \left[ 1 - \left( \frac{y}{h} \right)^2 \right] \quad (2.6)$$

where  $r$  and  $y$  are measured from the centerline,  $r_o$  is tube radius and  $h$  is the distance from the centerline to the plates. [Ref.1]

## 1. The Heat Transfer

The heat transfer for a duct flow can be expressed using Newton's law of cooling,  $q'' = h(T_w - T_m)$ , where  $h$  is the convection heat transfer coefficient,  $T_w$  is the wall temperature and  $T_m$  is the mean temperature of the fluid, where  $T_m$  is

$$T_m = \frac{\int \rho u c_v T dA}{\dot{m} c_v} \quad (2.7)$$

For the constant  $c_v$  and incompressible flow through the circular tube,  $T_m$  is

$$T_m = \frac{2}{V r_o^2} \int_0^{r_o} u T r dr \quad (2.8)$$

where for the flow between the parallel plates is

$$T_m = \frac{1}{V h} \int_0^h u T dy \quad (2.9)$$

$y$  and  $r$  are measured from the centerline,  $r_o$  is the radius of the tube and  $h$  is the distance from the centerline to the plate. [Ref. 1]

The energy equations for the cylindrical and rectangular coordinates are

$$u\rho \frac{\partial i}{\partial x} + v_r \rho \frac{\partial i}{\partial r} - \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( k r \frac{\partial t}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \phi} \left( k \frac{\partial t}{\partial \phi} \right) + \frac{\partial}{\partial x} \left( k \frac{\partial t}{\partial x} \right) \right] = 0 \quad (2.10)$$

and

$$u\rho \frac{\partial i}{\partial x} + v\rho \frac{\partial i}{\partial y} + w\rho \frac{\partial i}{\partial z} - \left[ \frac{\partial}{\partial x} \left( k \frac{\partial t}{\partial x} \right) + \frac{\partial}{\partial y} \left( k \frac{\partial t}{\partial y} \right) + \frac{\partial}{\partial z} \left( k \frac{\partial t}{\partial z} \right) \right] = 0 \quad (2.11)$$

where "i" is the enthalpy. These equations are applied to the problem including the boundary conditions and the temperature distribution is calculated analytically or numerically. [Ref. 2]

## 2. The Thermally Fully Developed Flow

In thermally fully developed case, the relative shape of the profile no longer changes although the temperature profile changes with x. The criterion for a fully developed temperature profile for a circular tube is

$$\frac{\partial}{\partial r} \left( \frac{T_w - T}{T_w - T_m} \right)_{r=r_0} = \text{const} \quad (2.12)$$

This condition is reached in a duct flow whether the case is uniform heat flux ( $q''$  through the wall is constant) or uniform wall temperature (The wall temperature is constant). These two cases have a lot of applications in engineering. For example, constant wall temperature in boiling or condensation, or constant heat flux using the electrical heater. In the thermally fully developed flow, the convection coefficient "h" is constant, independent of x. [Ref. 1]

## B. BACKGROUND STUDIES

The heat transfer solution for laminar fully developed parabolic velocity profile flow inside a circular tube in the thermal region and subject to uniform tube wall temperature was treated for the first time by Graetz in 1883 and is known in the heat transfer literature as Graetz problem. In his problem Graetz neglected the axial conduction and solved the problem. [Ref. 3]

The problem statement is:

$$\frac{1}{2}(1-r_*^2) \frac{\partial \theta_*}{\partial x_*} = \frac{\partial^2 \theta_*}{\partial x_*^2} + \frac{1}{r_*} \frac{\partial \theta_*}{\partial r_*} \quad (2.13)$$

$$\theta_* = 0 \quad \text{at } r_* = 1$$

$$\frac{\partial \theta}{\partial r_*} = 0 \quad \text{at } r_* = 0$$

$$\theta_* = 1 \quad \text{at } x_* = 1$$

where

$$\theta_* = \frac{T - T_0}{T_{in} - T_0}, \quad r_* = \frac{r}{r_0}, \quad x_* = \frac{x/D}{Re_D Pr}$$

and  $T_0$  is the wall temperature,  $T_{in}$  is the initial temperature,  $r$  is the radial distance,  $x$  is the axial distance,  $D$  is the diameter,  $Re_D$  is the Reynolds number and  $Pr$  is the Prandtl number. [Ref. 4]

The velocity profile is fully developed much before the temperature profile when the Prandtl number is high relative to one. For such situations, Graetz solution is reasonably well justified. But the Prandtl number is very small when the fluid is liquid metal and therefore the axial conduction effects may not be neglected. For such cases, since the axial conduction effects are not included to the result, the solution may give inaccurate results.

Michelsen and Villadsen investigated the Graetz problem with axial heat conduction for circular tube constant wall temperature case by using a numerical procedure. They used a method that is the combination of orthogonal collocation and matrix diagonalization. They didn't include the wall conduction effects in the problem.

The problem may be defined as:

$$(1 - x^2) \frac{\partial \theta}{\partial y} = \frac{1}{x} \frac{\partial}{\partial x} \left( x \frac{\partial \theta}{\partial x} \right) + \frac{1}{Pe^2} \frac{\partial^2 \theta}{\partial y^2} \quad (2.14)$$

where

$$Pe = \frac{2R \langle V_z \rangle \rho C_p}{k}, \quad x = \frac{r}{R}, \quad y = \frac{z}{PeR}, \quad \theta = \frac{T - T_0}{T_b - T_0}, \quad \bar{\theta} = \frac{\bar{T} - T_0}{T_b - T_0}$$

The boundary conditions are:

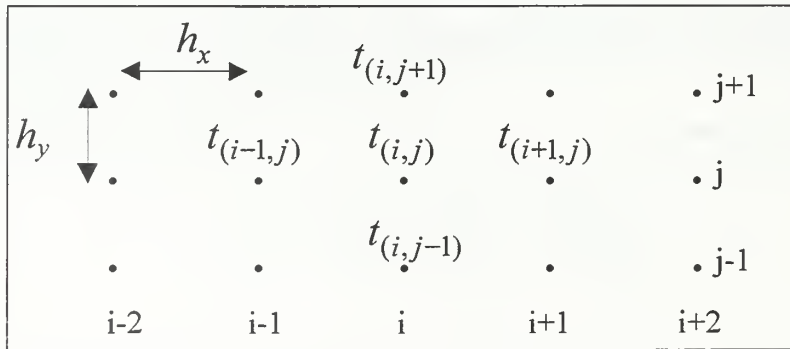
$$\begin{aligned} y \rightarrow -\infty \quad & \theta = 1 \\ x = 0 \quad & \frac{\partial \theta}{\partial x} = 0 \\ x = 1 \quad & \begin{cases} \frac{\partial \theta}{\partial x} = 0 & \text{for } y < 0 \\ \theta = 0 & \text{for } y \geq 0 \end{cases} \end{aligned}$$

where  $V_z$  is the average fluid velocity,  $R$  is the tube radius,  $z$  is the axial distance,  $r$  is the radial distance,  $T_b$  is the fluid temperature at  $z \rightarrow -\infty$ ,  $\bar{T}$  is the bulk mean temperature and  $T_0$  is the wall temperature at  $z \geq 0$ . They plotted the results, Nusselt number vs.  $y$ , heat flux vs.  $y$  and  $\bar{\theta}$  vs.  $y$  where the axial heat conduction effects are easily seen. But they didn't include the axial heat transfer inside the wall, which increases the fluid temperature very much upto the entrance as the wall conductivity gets larger. [Ref. 5]

X. Yin and H. H. Bau included the axial conduction effects of the wall to the internal flow through circular tube, by using two parameters, duct's outer/inner radius ratio and fluid/wall thermal conductivity ratio. They plotted the graphs of “*temperature vs. radius of the circular tube*” and “*Nusselt number vs. Peclet number*”, including the axial fluid and wall conduction effects. [Ref. 6]

## C. NUMERICAL METHOD

The governing equations are steady state elliptic partial differential equations. The temperature  $T(x,y)$  throughout the domain must satisfy the governing equation and the boundary conditions along the entire boundary. The finite difference method was used in this numerical study to express the governing equations numerically. The central, forward and backward difference expressions used in this numerical study for the grid points in figure(2) are as follows :



**Figure (2).** The grid points in finite difference methods

The central-difference expressions for the first and second derivatives with error order of  $h^2$  are:

$$\frac{\partial t}{\partial x} = \frac{t_{(i+1,j)} - t_{(i-1,j)}}{2h_x} \quad (2.15a)$$

$$\frac{\partial t}{\partial y} = \frac{t_{(i,j+1)} - t_{(i,j-1)}}{2h_y} \quad (2.15b)$$

$$\frac{\partial^2 t}{\partial x^2} = \frac{t_{(i+1,j)} + t_{(i-1,j)} - 2t_{(i,j)}}{(h_x)^2} \quad (2.16a)$$

$$\frac{\partial^2 t}{\partial y^2} = \frac{t_{(i,j+1)} + t_{(i,j-1)} - 2t_{(i,j)}}{(h_y)^2} \quad (2.16b)$$

The forward-difference expression with error order of  $h^2$  is:

$$\frac{\partial t}{\partial y} = \frac{-t_{(i,j+2)} + 4t_{(i,j+1)} - 3t_{(i,j)}}{2h_y} \quad (2.17)$$

The backward-difference expression with error order of  $h^2$  is:

$$\frac{\partial t}{\partial y} = \frac{-t_{(i,j+2)} + 4t_{(i,j+1)} - 3t_{(i,j)}}{2h_y}. \quad [\text{Ref. 7}] \quad (2.18)$$

Substituting these finite-difference expressions into the governing equations, the problem is reduced to a set of linear algebraic equations, depending on the number of grid points used. Usually large number of grid points is desirable, but the number of equations to be solved becomes too large in this case. Gauss Seidel iteration method can be used to obtain



a solution. All the temperatures of the grid points are then made equal to the initial temperature at the beginning of the numerical solution. The temperatures at the grid points are calculated by using the governing equations and the boundary conditions. After the temperature of a grid point is calculated using the finite difference equations, this updated temperature is used for the next grid and the temperature values converge to the exact temperature a little more in every iteration. A technique called over relaxation is used to speed the convergence of the Gauss Seidel method when applied to the elliptic partial differential equations. This technique is:

$$t_{(i,j\_new)} = wt_{(i,j\_new)} + (1 - w)t_{(i,j\_old)} \quad (2.19)$$

w is known as the relaxation parameter. For overrelaxation w is between 1 and 2; for underrelaxation between 0 and 1. For this method, overrelaxation must be used to speed up the convergence. [Ref. 7]

Forsythe and Wasow show that for a 45 by 45 mesh grid the optimum value of w is around 1.870. They also point out that when this relaxation parameter is used, the convergence is approximately 30 times faster than the usual process (w=1). [Ref. 8]

### III. GOVERNING EQUATIONS

#### A. DERIVATION OF THE EQUATIONS

##### 1. Flow Inside the Circular Tube, Constant Wall Temperature

###### Case

For flow in a circular tube, the viscous energy equation in cylindrical coordinate system is

$$u\rho\frac{\partial i}{\partial x} + v_r\rho\frac{\partial i}{\partial r} - \left[ \frac{1}{r}\frac{\partial}{\partial r}\left( kr\frac{\partial t}{\partial r} \right) + \frac{1}{r^2}\frac{\partial}{\partial \phi}\left( k\frac{\partial t}{\partial \phi} \right) + \frac{\partial}{\partial x}\left( k\frac{\partial t}{\partial x} \right) \right] = 0 \quad (3.1)$$

where  $di = cdt + \frac{1}{\rho}dP$  and  $v_r$  is the radial velocity [Ref.1]. Then the equation becomes

for the constant k

$$u\rho c\frac{\partial t}{\partial x} + v_r\rho c\frac{\partial t}{\partial r} - \left[ \frac{k}{r}\frac{\partial}{\partial r}\left( r\frac{\partial t}{\partial r} \right) + \frac{k}{r^2}\frac{\partial^2 t}{\partial \phi^2} + k\frac{\partial^2 t}{\partial x^2} \right] = 0 \quad (3.2)$$

The temperature distribution is symmetric  $(\partial^2 t / \partial \phi^2) = 0$  and the flow is hydrodynamically fully developed,  $v_r = 0$ . Equation then becomes

$$\left[ \frac{1}{r}\frac{\partial}{\partial r}\left( r\frac{\partial t}{\partial r} \right) + \frac{\partial^2 t}{\partial x^2} \right] = u\left( \frac{\rho c}{k} \right) \frac{\partial t}{\partial x}. \quad (3.3)$$

$\alpha = k/\rho c$ , the *thermal diffusivity* of the fluid. So the governing equation for the circular duct flow is

$$\frac{\partial^2 t}{\partial r^2} + \frac{1}{r}\frac{\partial t}{\partial r} + \frac{\partial^2 t}{\partial x^2} = \frac{u}{\alpha}\frac{\partial t}{\partial x} \quad (3.4)$$

Now, to non-dimensionalize the problem, assume that

$$\theta = \frac{t_0 - t}{t_0 - t_e}, \quad r^+ = \frac{r}{r_0}, \quad u^+ = \frac{u}{V}, \quad x^+ = \frac{x/r_0}{\text{Re Pr}}, \quad u^+ = 2(1 - r^{+2})$$

The governing equation becomes

$$\left(1 - r^{+2}\right) \frac{\partial \theta}{\partial x^+} = \frac{\partial^2 \theta}{\partial r^{+2}} + \frac{1}{r^+} \frac{\partial \theta}{\partial r^+} + \frac{1}{Pe^2} \frac{\partial^2 \theta}{\partial x^{+2}} \quad (3.5)$$

## 2. Flow Inside the Circular Tube, Constant Heat Flux Case

The governing equation for the circular duct flow is (3.4)

$$\frac{\partial^2 t}{\partial r^2} + \frac{1}{r} \frac{\partial t}{\partial r} + \frac{\partial^2 t}{\partial x^2} = \frac{u}{\alpha} \frac{\partial t}{\partial x}$$

Now, to non-dimensionalize the problem, assume that

$$\theta = \frac{t_e - t}{q_0'' d/k}, \quad r^+ = \frac{r}{r_0}, \quad u^+ = \frac{u}{V}, \quad x^+ = \frac{2x/d}{\text{Re}_d \text{Pr}}, \quad u^+ = 2(1 - r^{+2})$$

The governing equation becomes

$$\left(1 - r^{+2}\right) \frac{\partial \theta}{\partial x^+} = \frac{\partial^2 \theta}{\partial r^{+2}} + \frac{1}{r^+} \frac{\partial \theta}{\partial r^+} + \frac{1}{Pe^2} \frac{\partial^2 \theta}{\partial x^{+2}} \quad (3.6)$$

## 3. Flow Between the Parallel Plates, Constant Wall Temperature Case

The viscous energy equation in cartesian coordinate system for the flow between the parallel plates is

$$u\rho \frac{\partial i}{\partial x} + v\rho \frac{\partial i}{\partial y} + w\rho \frac{\partial i}{\partial z} - \left[ \frac{\partial}{\partial x} \left( k \frac{\partial t}{\partial x} \right) + \frac{\partial}{\partial y} \left( k \frac{\partial t}{\partial y} \right) + \frac{\partial}{\partial z} \left( k \frac{\partial t}{\partial z} \right) \right] = 0 \quad (3.7)$$

where  $di = cdt + \frac{1}{\rho} dP$  and  $v_r$  is the radial velocity [Ref.1]. Then the equation becomes

for the constant  $k$

$$u\rho c \frac{\partial t}{\partial x} + v\rho c \frac{\partial t}{\partial y} + w\rho c \frac{\partial t}{\partial z} - k \left[ \frac{\partial}{\partial x} \left( \frac{\partial t}{\partial x} \right) + \frac{\partial}{\partial y} \left( \frac{\partial t}{\partial y} \right) + \frac{\partial}{\partial z} \left( \frac{\partial t}{\partial z} \right) \right] = 0 \quad (3.8)$$

The temperature distribution on  $z$  direction is constant,  $\frac{\partial t}{\partial z} = 0$  and the flow is laminar and

hydrodynamically fully developed,  $v = 0$ ,  $w = 0$ . Equation then becomes

$$\left[ \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} \right] = u \left( \frac{\rho c}{k} \right) \frac{\partial t}{\partial x} \quad (3.9)$$

$\alpha = k/\rho c$ , the *thermal diffusivity* of the fluid. So the governing equation for the parallel plates flow is

$$\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} = \frac{u}{\alpha} \frac{\partial t}{\partial x} \quad (3.10)$$

Now, to non-dimensionalize the problem, assume that

$$\theta = \frac{t_0 - t}{t_0 - t_e}, \quad y^+ = \frac{y}{h}, \quad u^+ = \frac{u}{V}, \quad x^+ = \frac{2x/D_h}{\text{Re Pr}}, \quad u^+ = \frac{3}{2}(1 - y^{+2})$$

The governing equation becomes

$$\frac{1}{Pe_{Dh}^2} \frac{\partial^2 \theta}{\partial x^{+2}} + 4 \frac{\partial^2 \theta}{\partial y^{+2}} = \frac{u^+}{2} \frac{\partial \theta}{\partial x^+} \quad (3.11)$$

#### 4. Flow Between the Parallel Plates, Constant Heat Flux Case

The governing equation for the parallel plates flow is (3.10)

$$\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} = \frac{u}{\alpha} \frac{\partial t}{\partial x}$$

Now, to non-dimensionalize the problem, assume that

$$\theta = \frac{t_e - t}{q_0'' D_h / k}, \quad y^+ = \frac{y}{h}, \quad u^+ = \frac{u}{V}, \quad x^+ = \frac{2x / D_h}{\text{Re Pr}}, \quad u^+ = \frac{3}{2}(1 - y^{+2})$$

The governing equation becomes

$$\frac{1}{Pe_{Dh}^2} \frac{\partial^2 \theta}{\partial x^{+2}} + 4 \frac{\partial^2 \theta}{\partial y^{+2}} = \frac{u^+}{2} \frac{\partial \theta}{\partial x^+} \quad (3.12)$$

#### 5. Effects of Wall Conduction, Flow Inside the Circular Tube, Constant Wall Temperature

The governing equation for the circular duct flow is (3.4)

$$\frac{\partial^2 t}{\partial r^2} + \frac{1}{r} \frac{\partial t}{\partial r} + \frac{\partial^2 t}{\partial x^2} = \frac{u}{\alpha} \frac{\partial t}{\partial x}$$

Now, to non-dimensionalize the problem, assume that

$$\theta = \frac{t_0 - t}{t_0 - t_e}, \quad r^+ = \frac{r}{r_0}, \quad u^+ = \frac{u}{V}, \quad x^+ = \frac{x / r_0}{\text{Re Pr}}, \quad u^+ = 2(1 - r^{+2})$$

The governing equation becomes

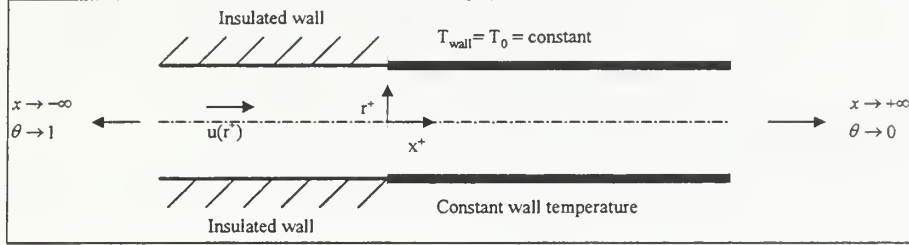
$$\left(1 - r^{+2}\right) \frac{\partial \theta}{\partial x^+} = \frac{\partial^2 \theta}{\partial r^{+2}} + \frac{1}{r^+} \frac{\partial \theta}{\partial r^+} + \frac{1}{Pe^2} \frac{\partial^2 \theta}{\partial x^{+2}} \quad (3.13)$$



## B. BOUNDARY CONDITIONS

### 1. Flow Inside the Circular Tube, Constant Wall Temperature

#### Case



**Figure 3.** The boundary conditions of circular tube, constant wall temperature flow

The governing equation is (3.5)

$$\left(1 - r^{+2}\right) \frac{\partial \theta}{\partial x^+} = \frac{\partial^2 \theta}{\partial r^{+2}} + \frac{1}{r^+} \frac{\partial \theta}{\partial r^+} + \frac{1}{Pe^2} \frac{\partial^2 \theta}{\partial x^{+2}}$$

where the parameters of the equation are defined above. The boundary conditions for this case is:

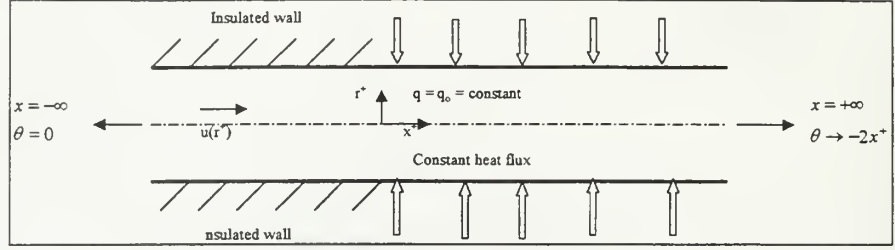
$$\text{at } r^+ = 0, \quad \frac{\partial \theta}{\partial r^+} = 0 \quad (\text{Symmetry boundary condition}) \quad (3.14a)$$

$$\text{at } r^+ = 1, \quad \begin{cases} \theta = 0 & \text{for } x^+ \geq 0 \\ \frac{\partial \theta}{\partial r^+} = 0 & \text{for } x^+ < 0 \end{cases} \quad (3.14b)$$

$$\text{as } x^+ \rightarrow -\infty, \quad t \rightarrow t_e, \quad \theta \rightarrow 1 \quad (3.14c)$$

$$\text{as } x^+ \rightarrow +\infty, \quad t \rightarrow t_0, \quad \theta \rightarrow 0 \quad (3.14d)$$

## 2. Flow Inside the Circular Tube, Constant Heat Flux Case



**Figure 4.** The boundary conditions of circular tube, constant heat flux flow

The governing equation is (3.6)

$$\left(1 - r^{+2}\right) \frac{\partial \theta}{\partial x^{+}} = \frac{\partial^2 \theta}{\partial r^{+2}} + \frac{1}{r^{+}} \frac{\partial \theta}{\partial r^{+}} + \frac{1}{Pe^2} \frac{\partial^2 \theta}{\partial x^{+2}}$$

where the parameters of the equation are defined above. The boundary conditions for this case is:

$$\text{at } r^{+}=0, \quad \frac{\partial \theta}{\partial r^{+}} = 0 \quad (\text{Symmetry boundary condition}) \quad (3.15a)$$

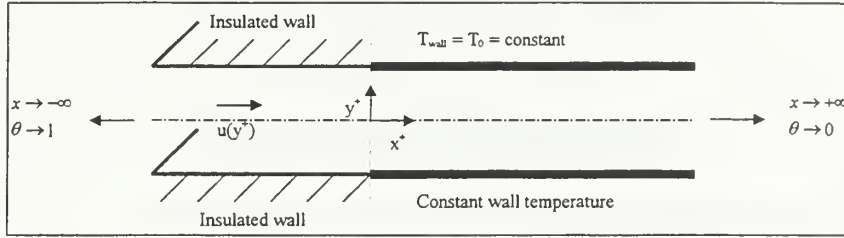
$$\text{at } r^{+}=1, \quad \begin{cases} \frac{\partial \theta}{\partial r^{+}} = -\frac{1}{2} & \text{for } x^{+} \geq 0 \\ \frac{\partial \theta}{\partial r^{+}} = 0 & \text{for } x^{+} < 0 \end{cases} \quad (3.15b)$$

$$\text{as } x^{+} \rightarrow -\infty, \quad t \rightarrow t_e, \quad \theta \rightarrow 0 \quad (3.15c)$$

$$\text{as } x^{+} \rightarrow +\infty, \quad \theta \rightarrow -2x^{+} + \left[ \frac{7}{48} + \frac{r^{+4}}{8} - \frac{r^{+2}}{2} \right] \quad (3.15d)$$

### 3. Flow Between the Parallel Plates, Constant Wall Temperature

#### Case



**Figure 5.** The boundary conditions of parallel plates, constant wall temperature flow

The governing equation is (3.11)

$$\frac{1}{Pe_{Dh}^2} \frac{\partial^2 \theta}{\partial x^{+2}} + 4 \frac{\partial^2 \theta}{\partial y^{+2}} = \frac{u^+}{2} \frac{\partial \theta}{\partial x^+}$$

where the parameters of the equation are defined above. The boundary conditions for this case is:

$$\text{at } y^+ = 0, \quad \frac{\partial \theta}{\partial y^+} = 0 \quad (\text{Symmetry boundary condition}) \quad (3.16a)$$

$$\text{at } y^+ = 1, \quad \begin{cases} \theta = 0 & \text{for } x^+ \geq 0 \\ \frac{\partial \theta}{\partial y^+} = 0 & \text{for } x^+ < 0 \end{cases} \quad (3.16b)$$

$$\text{as } x^+ \rightarrow -\infty, \quad t \rightarrow t_e, \quad \theta \rightarrow 1 \quad (3.16c)$$

$$\text{as } x^+ \rightarrow +\infty, \quad t \rightarrow t_0, \quad \theta \rightarrow 0 \quad (3.16d)$$

#### 4. Flow Between the Parallel Plates, Constant Heat Flux Case

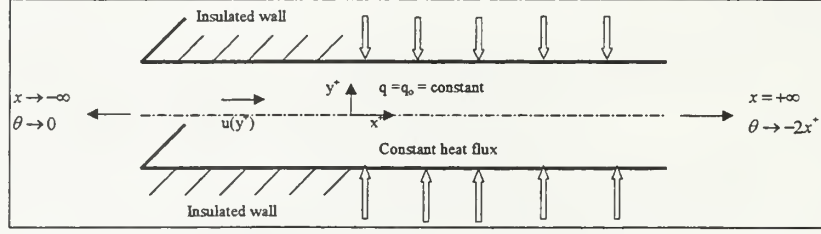


Figure 6. The boundary conditions of parallel plates, constant heat flux flow

The governing equation is (3.12)

$$\left(1 - r^{+2}\right) \frac{\partial \theta}{\partial x^{+}} = \frac{\partial^2 \theta}{\partial r^{+2}} + \frac{1}{r^{+}} \frac{\partial \theta}{\partial r^{+}} + \frac{1}{Pe^2} \frac{\partial^2 \theta}{\partial x^{+2}}$$

where the parameters of the equation are defined above. The boundary conditions for this case is:

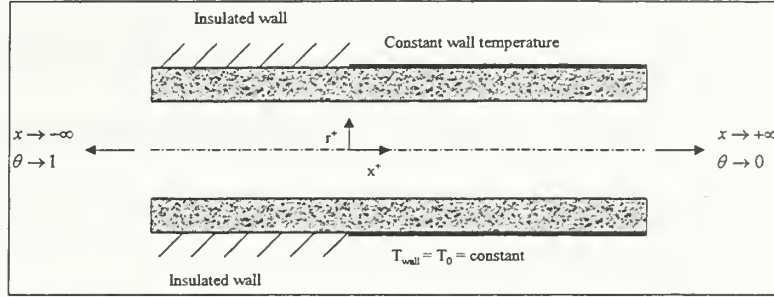
$$\text{at } y^{+}=0, \quad \frac{\partial \theta}{\partial r^{+}} = 0 \quad (\text{Symmetry boundary condition}) \quad (3.17a)$$

$$\text{at } y^{+}=1, \quad \begin{cases} \frac{\partial \theta}{\partial r^{+}} = -\frac{1}{4} & \text{for } x^{+} \geq 0 \\ \frac{\partial \theta}{\partial r^{+}} = 0 & \text{for } x^{+} < 0 \end{cases} \quad (3.17b)$$

$$\text{as } x^{+} \rightarrow -\infty, \quad t \rightarrow t_e, \quad \theta \rightarrow 0 \quad (3.17c)$$

$$\text{as } x^{+} \rightarrow +\infty, \quad t \rightarrow t_0, \quad \theta = -2x^{+} + \left[ \frac{39}{1120} + \frac{3}{16} \left( \frac{y^{+4}}{6} - y^{+2} \right) \right] \quad (3.17d)$$

## 5. Effects of Wall Conduction, Flow Inside the Circular Tube, Constant Wall Temperature



**Figure 7.** The boundary conditions of circular tube, constant wall temperature including the circular wall

The governing equation is (3.5)

$$\left(1 - r^{+2}\right) \frac{\partial \theta}{\partial x^+} = \frac{\partial^2 \theta}{\partial r^{+2}} + \frac{1}{r^+} \frac{\partial \theta}{\partial r^+} + \frac{1}{Pe^2} \frac{\partial^2 \theta}{\partial x^{+2}}$$

The boundary conditions for this case is:

$$\text{at } r^+ = 0, \quad \frac{\partial \theta}{\partial r^+} = 0 \quad (\text{Symmetry boundary condition}) \quad (3.18a)$$

$$\text{at } r^+ = 1, \quad \begin{cases} \theta_{wall} = \theta_{fluid} \\ \frac{\partial \theta_{wall}}{\partial r^+} = \frac{\partial \theta_{fluid}}{\partial r^+} \end{cases} \quad (3.18b)$$

$$\text{at } r^+ = 1 + \delta r, \quad \begin{cases} \frac{\partial \theta}{\partial r^+} = 0 & \text{for } x^+ \leq 0 \\ \theta = 0 & \text{for } x^+ > 0 \end{cases} \quad (3.18c)$$

where  $\delta r$  is  $\frac{r^*}{r_0}$ ,  $r^*$  is the thickness of the wall, and  $r_0$  is the inner radius of the tube.

$$\text{as } x^+ \rightarrow -\infty, \quad t \rightarrow t_e, \quad \theta \rightarrow 1 \quad (3.18d)$$

$$\text{as } x^+ \rightarrow +\infty, \quad t \rightarrow t_0, \quad \theta \rightarrow 0 \quad (3.18e)$$





## IV. NUMERICAL REPRESENTATION

### A. FLOW INSIDE THE CIRCULAR TUBE, CONSTANT WALL TEMPERATURE CASE

#### 1. The Governing Equation

The governing equation is (3.5)

$$\left(1 - r^{+2}\right) \frac{\partial \theta}{\partial x^+} = \frac{\partial^2 \theta}{\partial r^{+2}} + \frac{1}{r^+} \frac{\partial \theta}{\partial r^+} + \frac{1}{Pe_D^2} \frac{\partial^2 \theta}{\partial x^{+2}}$$

where

$$\theta = \frac{t_0 - t}{t_0 - t_e}, \quad r^+ = \frac{r}{r_0}, \quad x^+ = \frac{x/r_0}{Re Pr}.$$

The finite difference approximations for the first and second derivatives with error of order  $h^2$  are (figure 2.)

$$\frac{\partial \theta}{\partial x^+}_{(i,j)} = \frac{\theta_{(i+1,j)} - \theta_{(i-1,j)}}{2h_x} + o(h_x^2) \quad (4.1a)$$

$$\frac{\partial \theta}{\partial r^+}_{(i,j)} = \frac{\theta_{(i,j+1)} - \theta_{(i,j-1)}}{2h_y} + o(h_y^2) \quad (4.1b)$$

$$\frac{\partial^2 \theta}{\partial x^{+2}}_{(i,j)} = \frac{\theta_{(i+1,j)} + \theta_{(i-1,j)} - 2\theta_{(i,j)}}{h_x^2} + o(h_x^2) \quad (4.1c)$$

$$\frac{\partial^2 \theta}{\partial r^{+2}}_{(i,j)} = \frac{\theta_{(i,j+1)} + \theta_{(i,j-1)} - 2\theta_{(i,j)}}{h_y^2} + o(h_y^2) \quad (4.1d)$$

When these approximations are plugged into the governing equation (3.5), the equation becomes

$$\frac{1}{Pe_D^2} \left( \frac{\theta_{(i+1,j)} + \theta_{(i-1,j)} - 2\theta_{(i,j)}}{h_x^2} \right) + \left( \frac{\theta_{(i,j+1)} + \theta_{(i,j-1)} - 2\theta_{(i,j)}}{h_y^2} \right) + \frac{1}{y_{(i,j)}} \left( \frac{\theta_{(i,j+1)} - \theta_{(i,j-1)}}{2h_y} \right) - (1 - y_{(i,j)}^2) \left( \frac{\theta_{(i+1,j)} - \theta_{(i-1,j)}}{2h_x} \right) = 0 \quad (4.2)$$

After simplifying the expression (4.2),  $\theta_{(i,j)}$  becomes

$$\theta_{(i,j)} = \frac{\theta_{(i+1,j)} \left( \frac{1}{Pe_D^2 h_x^2} + \frac{y_{(i,j)}^2}{2h_x} - \frac{1}{2h_x} \right) + \theta_{(i-1,j)} \left( \frac{1}{Pe_D^2 h_x^2} - \frac{y_{(i,j)}^2}{2h_x} + \frac{1}{2h_x} \right) + \theta_{(i,j+1)} \left( \frac{1}{h_y^2} + \frac{1}{2h_y y_{(i,j)}} \right) + \theta_{(i,j-1)} \left( \frac{1}{h_y^2} - \frac{1}{2h_y y_{(i,j)}} \right)}{\frac{2}{Pe_D^2 h_x^2} + \frac{2}{h_y^2}} \quad (4.3)$$

## 2. The Boundary Conditions

The boundary conditions are:

### a) At the Center

$$\text{At } r^+ = 0, \quad \frac{\partial \theta}{\partial r^+} = 0 \quad (\text{Symmetry boundary condition})$$

When this boundary condition is applied to the governing equation, the element  $\frac{1}{r^+} \frac{\partial \theta}{\partial r^+}$

becomes  $\frac{0}{0}$ . After applying the L'Hospital's rule, the governing equation for the given

boundary condition becomes

$$\frac{\partial \theta}{\partial x^+} = 2 \frac{\partial^2 \theta}{\partial r^{+2}} + \frac{1}{Pe^2} \frac{\partial^2 \theta}{\partial x^{+2}} \quad (4.4)$$

Since the temperature distribution at the center on the radial direction is symmetric,

$$\theta_{(i,j+1)} = \theta_{(i,j-1)}.$$

After applying the finite difference method, the  $\theta_{(i,j)}$  at the boundary becomes

$$\theta_{(i,j)} = \frac{\theta_{(i+1,j)} \left( \frac{1}{Pe_D^2 h_x^2} - \frac{1}{2h_x} \right) + \theta_{(i-1,j)} \left( \frac{1}{Pe_D^2 h_x^2} + \frac{1}{2h_x} \right) + \theta_{(i,j+1)} \left( \frac{4}{h_y^2} \right)}{\frac{2}{Pe_D^2 h_x^2} + \frac{4}{h_y^2}} \quad (4.5)$$

#### b) At the Wall

$$\text{at } r^+ = 1, \quad \begin{cases} \theta_{(i,j)} = 0 & \text{for } x^+ \geq 0 \\ \frac{\partial \theta}{\partial r^+} = 0 & \text{for } x^+ < 0 \end{cases} \quad (4.6)$$

FOR  $x^+ < 0$

$$x^+ < 0, \quad r^+ = 1, \quad \frac{\partial \theta}{\partial r^+} = 0$$

The governing equation for the given boundary condition becomes

$$\frac{\partial^2 \theta}{\partial r^{+2}} + \frac{1}{Pe_D^2} \frac{\partial^2 \theta}{\partial x^{+2}} = 0 \quad (4.7)$$

Since the wall is insulated,  $\frac{\partial \theta}{\partial r^+} = 0$ ,  $\theta_{(i,j+1)} = \theta_{(i,j-1)}$  at the boundary.

After applying the finite difference approximations, the  $\theta_{(i,j)}$  at the boundary becomes

$$\theta_{(i,j)} = \frac{\theta_{(i+1,j)} \left( \frac{1}{Pe_D^2 h_x^2} \right) + \theta_{(i-1,j)} \left( \frac{1}{Pe_D^2 h_x^2} \right) + \theta_{(i,j-1)} \left( \frac{2}{h_y^2} \right)}{\frac{2}{Pe_D^2 h_x^2} + \frac{2}{h_y^2}} \quad (4.8)$$

c) As  $x \rightarrow -\infty$

$$\text{as } x^+ \rightarrow -\infty, \quad \theta_{(i,j)} \rightarrow 1 \quad (4.9)$$

d) As  $x \rightarrow +\infty$

$$\text{as } x^+ \rightarrow +\infty, \quad \theta_{(i,j)} \rightarrow 0 \quad (4.10)$$

### 3. The Bulk Mean Temperature

The mean temperature at a position “i” is (2.8)

$$t_{m(x)} = \frac{2}{Vr_0^2} \int_0^{r_0} u_{(x,r)} t_{(x,r)} r dr$$

where V is the average velocity and  $r_0$  is the radius of the tube. When  $\int_0^{r_0} u_{(x,r)} 2\pi r dr$  is

subtracted from both sides and dimensionless parameter  $r^+$  is substituted inside the equation, the equation becomes

$$(t_m - t_o) = \int_0^1 (t - t_o) u^+ 2r^+ dr^+$$

$$\text{Let's define the bulk mean temperature, } \theta_m = \frac{(t_m - t_o)}{(t_e - t_o)}, \quad (4.11a)$$

where  $t_e$  is the inlet temperature (as  $x \rightarrow -\infty$ ) and  $t_o$  is the constant wall temperature. (Fluid exit temperature as  $x \rightarrow +\infty$ )

Then the bulk mean temperature  $\theta_m$  becomes

$$\theta_{m(x^+)} = 4 \int_0^1 (1 - r^{+2}) \theta_{(x^+, r^+)} r^+ dr^+ \quad (4.11b)$$

$\theta$  is found out from the numerical calculation. After that  $\theta_m$  is calculated by solving the equation (4.11) numerically.

#### 4. The Nusselt Number

The Nusselt number is (2.3)

$$Nu = \frac{hd}{k}.$$

When the heat transfer equations  $q_o'' = h(T_o - T_m)$  and  $q_o'' = -k \frac{\partial \theta}{\partial r} \Big|_{r=R}$ , and the dimensionless parameter  $r^+$  are plugged in (2.3), the equation becomes

$$Nu = - \frac{2}{\theta_m} \frac{\partial \theta}{\partial r^+} \Big|_{r^+=1}. \quad (4.12)$$

The finite difference approximation for the equation (4.12) using the backward difference expression with error of order  $h^2$  becomes

$$Nu_{(i)} = \frac{2}{\theta_{m(i)}} \left( \frac{-3\theta_{(i,j)} + 4\theta_{(i,j-1)} - \theta_{(i,j-2)}}{2h_y} \right) \quad (4.13)$$

where “j” is the value which corresponds to  $r^+=1$ .

## 5. Heat Absorption Upto the Entrance

Assume a differential volume,  $dV = 2\pi r dr$  between  $x = -\infty$  and  $x = 0$ . The energy stored in this control volume is

$$dE = c_p \Delta t dm \quad (4.14)$$

where  $\Delta t$  is the temperature difference between entrance temperature and the steady state temperature after the heating starts,  $\Delta t = t_{(x,r)} - t_e$  and  $dm$  is the mass of the control volume,  $dm = \rho dV$ . The energy absorbed in the control volume becomes

$$E_{absorbed} = 2\pi \rho c_p \int_{-\infty}^0 \int_0^{r_o} (t - t_e) r dr dx. \quad (4.15)$$

When  $2\pi \rho c_p \int_{-\infty}^0 \int_0^{r_o} t_o r dr dx$  is subtracted from both sides and  $(t_e - t_o)$  is divided to both sides, the equation (4.15) becomes

$$\frac{E_{absorbed}}{(t_e - t_o) \rho c_p} = 2\pi \int_{-\infty}^0 \int_0^{r_o} (\theta - 1) r dr dx \quad (4.16)$$

where  $\theta = \frac{t - t_o}{t_e - t_o}$ . As the non-dimensional parameters of  $x^+ = \frac{x/r_o}{Re Pr}$  and  $r^+ = \frac{r}{r_o}$  and

their first derivatives are substituted into the equation (4.16), the equation becomes

$$\frac{E_{absorbed}}{(t_o - t_e) \rho c_p r_o^3} = -2\pi Pe \int_{-\infty}^0 \int_0^1 (\theta - 1) r^+ dr^+ dx^+ \quad (4.17)$$

## B. FLOW INSIDE THE CIRCULAR TUBE, CONSTANT HEAT FLUX CASE

### 1. The Governing Equation

The governing equation is (3.6)

$$\left(1 - r^{+2}\right) \frac{\partial \theta}{\partial x^+} = \frac{\partial^2 \theta}{\partial r^{+2}} + \frac{1}{r^+} \frac{\partial \theta}{\partial r^+} + \frac{1}{Pe_D^2} \frac{\partial^2 \theta}{\partial x^{+2}}$$

where

$$\theta = \frac{t_e - t}{q_0'' d/k}, \quad r^+ = \frac{r}{r_0}, \quad x^+ = \frac{2x/d}{Re_d Pr}.$$

The finite difference approximations for the first and second derivatives with the error of order  $h^2$  are (4.1a,b,c,d) (figure 2.)

$$\frac{\partial \theta}{\partial x^+}_{(i,j)} = \frac{\theta_{(i+1,j)} - \theta_{(i-1,j)}}{2h_x} + o(h_x^2)$$

$$\frac{\partial \theta}{\partial r^+}_{(i,j)} = \frac{\theta_{(i,j+1)} - \theta_{(i,j-1)}}{2h_y} + o(h_y^2)$$

$$\frac{\partial^2 \theta}{\partial x^{+2}}_{(i,j)} = \frac{\theta_{(i+1,j)} + \theta_{(i-1,j)} - 2\theta_{(i,j)}}{h_x^2} + o(h_x^2)$$

$$\frac{\partial^2 \theta}{\partial r^{+2}}_{(i,j)} = \frac{\theta_{(i,j+1)} + \theta_{(i,j-1)} - 2\theta_{(i,j)}}{h_y^2} + o(h_y^2)$$



When these approximations are plugged into the governing equation (3.6), the equation becomes

$$\frac{1}{Pe_D^2} \left( \frac{\theta_{(i+1,j)} + \theta_{(i-1,j)} - 2\theta_{(i,j)}}{h_x^2} \right) + \left( \frac{\theta_{(i,j+1)} + \theta_{(i,j-1)} - 2\theta_{(i,j)}}{h_y^2} \right) + \frac{1}{y_{(i,j)}} \left( \frac{\theta_{(i,j+1)} - \theta_{(i,j-1)}}{2h_y} \right) - (1 - y_{(i,j)}^2) \left( \frac{\theta_{(i+1,j)} - \theta_{(i-1,j)}}{2h_x} \right) = 0 \quad (4.18)$$

After simplifying the expression (4.18),  $\theta_{(i,j)}$  becomes

$$\theta_{(i,j)} = \frac{\theta_{(i+1,j)} \left( \frac{1}{Pe_D^2 h_x^2} + \frac{y_{(i,j)}^2}{2h_x} - \frac{1}{2h_x} \right) + \theta_{(i-1,j)} \left( \frac{1}{Pe_D^2 h_x^2} - \frac{y_{(i,j)}^2}{2h_x} + \frac{1}{2h_x} \right) + \theta_{(i,j+1)} \left( \frac{1}{h_y^2} + \frac{1}{2h_y y_{(i,j)}} \right) + \theta_{(i,j-1)} \left( \frac{1}{h_y^2} - \frac{1}{2h_y y_{(i,j)}} \right)}{\frac{2}{Pe_D^2 h_x^2} + \frac{2}{h_y^2}} \quad (4.19)$$

## 2. The Boundary Conditions

The boundary conditions are:

### a) At the Center

$$\text{At } r^+ = 0, \quad \frac{\partial \theta}{\partial r^+} = 0 \quad (\text{Symmetry boundary condition})$$

When this boundary condition is applied to the governing equation, the element  $\frac{1}{r^+} \frac{\partial \theta}{\partial r^+}$  becomes  $\frac{0}{0}$ . After applying the L'Hospital's rule, the governing equation for the given

boundary condition becomes

$$\frac{\partial \theta}{\partial x^+} = 2 \frac{\partial^2 \theta}{\partial r^{+2}} + \frac{1}{Pe^2} \frac{\partial^2 \theta}{\partial x^{+2}} \quad (4.20)$$

Since the temperature distribution at the center on the radial direction is symmetric,

$$\theta_{(i,j+1)} = \theta_{(i,j-1)}.$$

After applying the finite difference method, the  $\theta_{(i,j)}$  at the boundary becomes

$$\theta_{(i,j)} = \frac{\theta_{(i+1,j)} \left( \frac{1}{Pe_D^2 h_x^2} - \frac{1}{2h_x} \right) + \theta_{(i-1,j)} \left( \frac{1}{Pe_D^2 h_x^2} + \frac{1}{2h_x} \right) + \theta_{(i,j+1)} \left( \frac{4}{h_y^2} \right)}{\frac{2}{Pe_D^2 h_x^2} + \frac{2}{h_y^2}} \quad (4.21)$$

### b) At the Wall

$$\text{at } r^+ = 1, \quad \frac{\partial \theta}{\partial r^+} = \begin{cases} -\frac{1}{2} & \text{for } x^+ \geq 0 \\ 0 & \text{for } x^+ < 0 \end{cases} \quad (4.22)$$

FOR  $x^+ \geq 0$ ,

$$k \frac{\partial t}{\partial r} = q_o'', \quad \theta(x^+, r^+) = \frac{t_e - t_{(x,r)}}{q_o'' d/k}, \quad r^+ = \frac{r}{r_o}$$

so the equation becomes  $\frac{\partial \theta}{\partial r^+} = -\frac{1}{2}$ .

The finite difference approximation for this equation is  $\frac{\theta_{(i,j+1)} - \theta_{(i,j-1)}}{2h_y} = -\frac{1}{2}$ .

When the boundary condition is applied to the governing equation (3.6), the equation becomes

$$\frac{\partial^2 \theta}{\partial r^{+2}} - \frac{1}{2r^+} + \frac{1}{Pe^2} \frac{\partial^2 \theta}{\partial x^{+2}} = 0 \quad (4.23)$$

As the finite difference approximations are substituted in the equation above, the equation becomes

$$\theta_{(i,j)} = \frac{\theta_{(i+1,j)} \left( \frac{1}{Pe_D^2 h_x^2} \right) + \theta_{(i-1,j)} \left( \frac{1}{Pe_D^2 h_x^2} \right) + \theta_{(i,j-1)} \left( \frac{2}{h_y^2} \right) - \frac{1}{h_y} - \frac{1}{2}}{\frac{2}{Pe_D^2 h_x^2} + \frac{2}{h_y^2}} \quad (4.24)$$

FOR  $x^+ < 0$

$$x^+ < 0, \quad r^+ = 1, \quad \frac{\partial \theta}{\partial r^+} = 0$$

The governing equation for the given boundary condition becomes

$$\frac{\partial^2 \theta}{\partial r^{+2}} + \frac{1}{Pe_D^2} \frac{\partial^2 \theta}{\partial x^{+2}} = 0 \quad (4.25)$$

Since the wall is insulated,  $\frac{\partial \theta}{\partial r^+} = 0$ ,  $\theta_{(i,j+1)} = \theta_{(i,j-1)}$  at the boundary.

After applying the finite difference approximations to (4.25), the  $\theta_{(i,j)}$  at the boundary becomes

$$\theta_{(i,j)} = \frac{\theta_{(i+1,j)} \left( \frac{1}{Pe_D^2 h_x^2} \right) + \theta_{(i-1,j)} \left( \frac{1}{Pe_D^2 h_x^2} \right) + \theta_{(i,j-1)} \left( \frac{2}{h_y^2} \right)}{\frac{2}{Pe_D^2 h_x^2} + \frac{2}{h_y^2}} \quad (4.26)$$

**c) As  $x \rightarrow -\infty$**

$$\text{as } x^+ \rightarrow -\infty, \quad \theta_{(i,j)} \rightarrow 0 \quad (4.27)$$

d) As  $x \rightarrow +\infty$

$$\text{as } x^+ \rightarrow +\infty, \quad \theta_{(i,j)} \rightarrow -2x^+ + \left( \frac{7}{48} + \frac{r^{+4}}{8} - \frac{r^{+2}}{2} \right) \quad (4.28)$$

### 3. The Bulk Mean Temperature

The mean temperature at a position “i” is (2.8)

$$t_m = \frac{2}{Vr_0^2} \int_0^{r_0} utr dr$$

where  $V$  is the average velocity and  $r_0$  is the radius of the tube. When  $\int_0^{r_0} t_e u 2\pi r dr$  is subtracted from both sides and dimensionless parameter  $r^+$  is substituted inside the equation, the equation becomes

$$(t_m - t_e) = \int_0^1 (t - t_e) u^+ 2r^+ dr^+ \quad (4.29)$$

Let's define the bulk mean temperature,

$$\theta_m = \frac{(t_e - t_m)}{q_o'' d/k}, \quad (4.30a)$$

where  $t_e$  is the inlet temperature (as  $x \rightarrow -\infty$ ).

Then the bulk mean temperature  $\theta_m$  becomes

$$\theta_m = 4 \int_0^1 (1 - r^{+2}) \theta r^+ dr^+ \quad (4.30b)$$

$\theta$  is found out from the numerical calculation. After that  $\theta_m$  is calculated by solving the equation above numerically.

#### 4. The Nusselt Number

The Nusselt number:  $Nu = \frac{hd}{k}$ .

The heat transfer from the wall:  $q_o'' = h(t_{(x^+, r^+=1)} - t_m)$ .

The temperature at the position of  $(x^+, r^+=1)$ :  $t_{(x^+, r^+=1)} = t_e - \theta q_o'' d/k$

The average temperature at  $(x^+)$ :  $t_m = t_e - \theta_m q_o'' d/k$

When these equations are substituted into the Nusselt number equation, Nusselt number becomes:

$$Nu = \frac{1}{\theta_m - \theta} \quad (4.31)$$

#### 5. Heat Absorption Upto the Entrance

Assume a differential volume,  $dV = 2\pi r dr$  between  $x = -\infty$  and  $x = 0$ . The energy stored in this control volume is (4.14)

$$dE = c_p \Delta t dm$$

where  $\Delta t$  is the temperature difference between entrance temperature and the steady state temperature after the heating starts,  $\Delta t = t_{(x,r)} - t_e$  and  $dm$  is the mass of the control volume,  $dm = \rho dV$ . The energy absorbed in the control volume becomes

$$E_{absorbed} = 2\pi\rho c_p \int_{-\infty}^0 \int_0^{r_o} (t - t_e) r dr dx. \quad (4.32)$$

When both sides are divided by  $q_o'' d/k$ , the equation (4.32) becomes

$$\frac{E_{absorbed}}{q_o'' \rho c_p r_o / k} = -4\pi \int_{-\infty}^0 \int_0^{r_o} \theta r dr dx \quad (4.33)$$

where  $\theta = \frac{t_e - t}{q_o'' d/k}$ . As the non-dimensional parameters of  $x^+ = \frac{x/r_o}{\text{Re Pr}}$  and  $r^+ = \frac{r}{r_o}$  and

their first derivatives are substituted into the equation (4.33), the equation becomes

$$\frac{E_{absorbed}}{q_o'' \rho c_p r_o^4 / k} = -4\pi Pe \int_{-\infty}^0 \int_0^1 \theta r^+ dr^+ dx^+ \quad (4.34)$$

## C. FLOW BETWEEN THE PARALLEL PLATES, CONSTANT WALL TEMPERATURE CASE

### 1. The Governing Equation

The governing equation is defined in equation (3.11) as

$$\frac{1}{Pe_{Dh}^2} \frac{\partial^2 \theta}{\partial x^{+2}} + 4 \frac{\partial^2 \theta}{\partial y^{+2}} = \frac{u^+}{2} \frac{\partial \theta}{\partial x^+}$$

where

$$\theta = \frac{t_0 - t}{t_0 - t_e}, \quad y^+ = \frac{y}{h}, \quad x^+ = \frac{2x/D_h}{\text{Re Pr}}, \quad D_h = 4h.$$

The finite difference approximations for the first and second derivatives with error of order  $h^2$  are (4.1a,b,c,d) (figure 2.)

$$\frac{\partial \theta}{\partial x^+}_{(i,j)} = \frac{\theta_{(i+1,j)} - \theta_{(i-1,j)}}{2h_x} + o(h_x^2)$$

$$\frac{\partial \theta}{\partial r^+}_{(i,j)} = \frac{\theta_{(i,j+1)} - \theta_{(i,j-1)}}{2h_y} + o(h_y^2)$$

$$\frac{\partial^2 \theta}{\partial x^{+2}}_{(i,j)} = \frac{\theta_{(i+1,j)} + \theta_{(i-1,j)} - 2\theta_{(i,j)}}{h_x^2} + o(h_x^2)$$

$$\frac{\partial^2 \theta}{\partial r^{+2}}_{(i,j)} = \frac{\theta_{(i,j+1)} + \theta_{(i,j-1)} - 2\theta_{(i,j)}}{h_y^2} + o(h_y^2)$$

When these approximations and  $u^+$  are plugged into the governing equation (3.11),  $\theta_{(i,j)}$

becomes

$$\theta_{(i,j)} = \frac{\theta_{(i+1,j)} \left( \frac{1}{Pe_{D_h}^2 h_x^2} + \frac{3y_{(i,j)}^2}{8h_x} - \frac{3}{8h_x} \right) + \theta_{(i-1,j)} \left( \frac{1}{Pe_{D_h}^2 h_x^2} - \frac{3y_{(i,j)}^2}{8h_x} + \frac{3}{8h_x} \right) + \theta_{(i,j+1)} \left( \frac{4}{h_y^2} \right) + \theta_{(i,j-1)} \left( \frac{4}{h_y^2} \right)}{\frac{2}{Pe_{D_h}^2 h_x^2} + \frac{8}{h_y^2}} \quad (4.35)$$

## 2. The Boundary Conditions

The boundary conditions are:

### a) At the Center

$$\text{At } y^+=0, \quad j=1, \quad \frac{\partial \theta}{\partial y^+} = 0 \quad (\text{Symmetry boundary condition})$$



When this boundary condition is applied to the governing equation, the governing equation becomes

$$\frac{1}{Pe_{D_h}^2} \frac{\partial^2 \theta}{\partial x^{+2}} + 4 \frac{\partial^2 \theta}{\partial y^{+2}} = \frac{3}{4} \frac{\partial \theta}{\partial x^+} \quad (4.36)$$

Since the temperature distribution at the center on the radial direction is symmetric,

$$\theta_{(i,j+1)} = \theta_{(i,j-1)}$$

at the center.

After applying the finite difference method to (4.36), the  $\theta_{(i,j)}$  at the boundary becomes

$$\theta_{(i,1)} = \frac{\theta_{(i+1,1)} \left( \frac{1}{Pe_D^2 h_x^2} - \frac{3}{8h_x} \right) + \theta_{(i-1,1)} \left( \frac{1}{Pe_D^2 h_x^2} + \frac{3}{8h_x} \right) + \theta_{(i,2)} \left( \frac{8}{h_y^2} \right)}{\frac{2}{Pe_D^2 h_x^2} + \frac{8}{h_y^2}} \quad (4.37)$$

#### b) At the Wall

$$\text{at } y^+ = 1, \quad \begin{cases} \theta_{(i,j)} = 0 & \text{for } x^+ \geq 0 \\ \frac{\partial \theta}{\partial y^+} = 0 & \text{for } x^+ < 0 \end{cases} \quad (4.38)$$

FOR  $\underline{x^+ < 0}$

$$x^+ < 0, \quad y^+ = 1, \quad \frac{\partial \theta}{\partial y^+} = 0$$

The governing equation for the given boundary condition becomes

$$\frac{1}{Pe_{D_h}^2} \frac{\partial^2 \theta}{\partial x^{+2}} + 4 \frac{\partial^2 \theta}{\partial y^{+2}} = 0 \quad (4.39)$$

Since the wall is insulated,  $\frac{\partial \theta}{\partial y^+} = 0$ ,  $\theta_{(i,j+1)} = \theta_{(i,j-1)}$  at the boundary.

After applying the finite difference approximations, the  $\theta_{(i,j)}$  at the boundary becomes

$$\theta_{(i,j)} = \frac{\theta_{(i+1,j)} \left( \frac{1}{Pe_D^2 h_x^2} \right) + \theta_{(i-1,j)} \left( \frac{1}{Pe_D^2 h_x^2} \right) + \theta_{(i,j-1)} \left( \frac{8}{h_y^2} \right)}{\frac{2}{Pe_D^2 h_x^2} + \frac{8}{h_y^2}} \quad (4.40)$$

**e) As  $x \rightarrow -\infty$**

$$\text{as } x^+ \rightarrow -\infty, \quad \theta_{(i,j)} \rightarrow 1 \quad (4.41)$$

**f) As  $x \rightarrow +\infty$**

$$\text{as } x^+ \rightarrow +\infty, \quad \theta_{(i,j)} \rightarrow 0 \quad (4.42)$$

### 3. The Bulk Mean Temperature

The mean temperature is defined in equation (2.9) as

$$t_m = \frac{1}{Vh} \int_0^h u t dy.$$

When the dimensionless parameters  $u^+$ ,  $y^+$  are substituted in (2.9), the  $t_m$  becomes

$$t_m = \int_0^1 u^+ t dy^+$$

where  $V$  is the average velocity and  $h$  is the distance from the centerline to the plate.

When  $t_o = \int_0^1 t_o u^+ dy^+$  is subtracted from both sides, the equation becomes

$$(t_o - t_m) = \int_0^1 (t_o - t) u^+ dy^+$$

Let's define the bulk mean temperature,

$$\theta_m = \frac{(t_o - t_m)}{(t_o - t_e)}, \quad (4.43a)$$

where  $t_e$  is the inlet temperature (as  $x \rightarrow -\infty$ ) and  $t_o$  is the constant wall temperature. (Fluid exit temperature as  $x \rightarrow +\infty$ )

Then the bulk mean temperature  $\theta_m$  becomes

$$\theta_m = \frac{3}{2} \int_0^1 (1 - y^{+2}) \theta dy^+. \quad (4.43b)$$

$\theta$  is taken from the numerical calculation. After that  $\theta_m$  is calculated by solving the equation (4.43) numerically.

#### 4. The Nusselt Number

The Nusselt number is defined as (2.3)

$$Nu = \frac{hD_h}{k}.$$

When the heat transfer equations  $q_o'' = h(T_o - T_m)$ ,  $q_o'' = k \frac{\partial \theta}{\partial y} \Big|_{y=h}$ , and the dimensionless

parameter  $r^+$  are plugged in (2.3), the equation becomes

$$Nu = -\frac{4}{\theta_m} \frac{\partial \theta}{\partial y^+} \Big|_{y^+=1}. \quad (4.44)$$

The finite difference approximation for the equation (4.44) using the backward difference expression with error of order  $h^2$  becomes

$$Nu_i = -\frac{4}{\theta_{m_i}} \left( \frac{3\theta_{(i,j)} - 4\theta_{(i,j-1)} + \theta_{(i,j-2)}}{2h_y} \right) \quad (4.45)$$

## 5. Heat Absorption Upto the Entrance

Assume a differential volume,  $dV = bdx dy$  between  $x = -\infty$  and  $x = 0$ . The energy stored in this control volume is (4.14)

$$dE = c_p \Delta t dm$$

where  $\Delta t$  is the temperature difference between entrance temperature and the steady state temperature after the heating starts,  $\Delta t = t_{(x,y)} - t_e$  and  $dm$  is the mass of the control volume,  $dm = \rho dV$ . The energy absorbed in the control volume becomes

$$E_{absorbed} = b\rho c_p \int_{-\infty}^0 \int_{-h}^{+h} (t - t_e) dy dx. \quad (4.46)$$

When  $2b\rho c_p \int_{-\infty}^0 \int_0^h t_o dy dx$  is subtracted from both sides and  $(t_e - t_o)$  is divided to both

sides, the equation (4.46) becomes

$$\frac{E_{absorbed}}{b\rho c_p (t_e - t_o)} = 2 \int_{-\infty}^0 \int_0^h (\theta - 1) dy dx \quad (4.47)$$

where  $\theta = \frac{t - t_o}{(t_e - t_o)}$ . As the non-dimensional parameters of  $x^+ = \frac{2x/D_h}{\text{Re Pr}}$  and

$r^+ = \frac{r}{r_o}$  and their first derivatives are substituted into the equation (4.47), the equation

becomes

$$\frac{E_{\text{absorbed}}}{h^2 b \rho c_p (t_o - t_e)} = -4Pe \int_{-\infty}^0 \int_0^1 (\theta - 1) dy^+ dx^+ \quad (4.48)$$

## D. FLOW BETWEEN THE PARALLEL PLATES, CONSTANT HEAT FLUX CASE

### 1. The Governing Equation

The governing equation is (3.12)

$$\frac{1}{Pe_{D_h}^2} \frac{\partial^2 \theta}{\partial x^{+2}} + 4 \frac{\partial^2 \theta}{\partial y^{+2}} = \frac{u^+}{2} \frac{\partial \theta}{\partial x^+}$$

where

$$\theta = \frac{t_e - t}{q_o'' D_h / k}, \quad y^+ = \frac{y}{h}, \quad x^+ = \frac{2x / D_h}{\text{Re Pr}}, \quad D_h = 4h$$

The finite difference approximations for the first and second derivatives are (4.1a,b,c,d)

$$\frac{\partial \theta}{\partial x^+}_{(i,j)} = \frac{\theta_{(i+1,j)} - \theta_{(i-1,j)}}{2h_x} + o(h_x^2)$$

$$\frac{\partial \theta}{\partial r^+}_{(i,j)} = \frac{\theta_{(i,j+1)} - \theta_{(i,j-1)}}{2h_y} + o(h_y^2)$$

$$\frac{\partial^2 \theta}{\partial x^{+2}}_{(i,j)} = \frac{\theta_{(i+1,j)} + \theta_{(i-1,j)} - 2\theta_{(i,j)}}{h_x^2} + o(h_x^2)$$

$$\frac{\partial^2 \theta}{\partial r^{+2}}_{(i,j)} = \frac{\theta_{(i,j+1)} + \theta_{(i,j-1)} - 2\theta_{(i,j)}}{h_y^2} + o(h_y^2)$$

When these approximations and  $u^+$  are plugged into the governing equation, the bulk temperature at the position (i ,j) is written as

$$\theta_{(i,j)} = \frac{\theta_{(i+1,j)} \left( \frac{1}{Pe_D^2 h_x^2} + \frac{3y_{(i,j)}^2}{8h_x} - \frac{3}{8h_x} \right) + \theta_{(i-1,j)} \left( \frac{1}{Pe_D^2 h_x^2} - \frac{3y_{(i,j)}^2}{8h_x} + \frac{3}{8h_x} \right) + \theta_{(i,j+1)} \left( \frac{4}{h_y^2} \right) + \theta_{(i,j-1)} \left( \frac{4}{h_y^2} \right)}{\frac{2}{Pe_D^2 h_x^2} + \frac{8}{h_y^2}} \quad (4.49)$$

## 2. The Boundary Conditions

The boundary conditions are:

### a) At the Center

$$\text{At } y^+=0, \quad j=1, \quad \frac{\partial \theta}{\partial y^+} = 0 \quad (\text{Symmetry boundary condition})$$

When this boundary condition is applied to the governing equation (3.12), the governing equation becomes

$$\frac{1}{Pe_{D_h}^2} \frac{\partial^2 \theta}{\partial x^{+2}} + 4 \frac{\partial^2 \theta}{\partial y^{+2}} = \frac{3}{4} \frac{\partial \theta}{\partial x^+} \quad (4.50)$$

Since the temperature distribution at the center on the radial direction is symmetric,

$$\theta_{(i,j+1)} = \theta_{(i,j-1)} \text{ at the center.}$$

After applying the finite difference method, the  $\theta_{(i,j)}$  at the boundary becomes

$$\theta_{(i,1)} = \frac{\theta_{(i+1,1)} \left( \frac{1}{Pe_D^2 h_x^2} - \frac{3}{8h_x} \right) + \theta_{(i-1,1)} \left( \frac{1}{Pe_D^2 h_x^2} + \frac{3}{8h_x} \right) + \theta_{(i,j+1)} \left( \frac{8}{h_y^2} \right)}{\frac{2}{Pe_D^2 h_x^2} + \frac{8}{h_y^2}} \quad (4.51)$$

**b) At the Wall**

$$\text{at } y^+=1, \quad \frac{\partial \theta}{\partial y^+} = \begin{cases} 0 & \text{for } x^+ < 0 \\ -\frac{1}{4} & \text{for } x^+ \geq 0 \end{cases}$$

FOR  $x^+ < 0$

$$x^+ < 0, \quad y^+ = 1, \quad \frac{\partial \theta}{\partial y^+} = 0$$

The governing equation for the given boundary condition becomes

$$\frac{1}{Pe_D^2} \frac{\partial^2 \theta}{\partial x^{+2}} + 4 \frac{\partial^2 \theta}{\partial y^{+2}} = 0 \quad (4.52)$$

Since the wall is insulated,  $\frac{\partial \theta}{\partial y^+} = 0$ ,  $\theta_{(i,j+1)} = \theta_{(i,j-1)}$  at the boundary.

After applying the finite difference approximations, the  $\theta_{(i,j)}$  at the boundary becomes

$$\theta_{(i,j)} = \frac{\theta_{(i+1,j)} \left( \frac{1}{Pe_D^2 h_x^2} \right) + \theta_{(i-1,j)} \left( \frac{1}{Pe_D^2 h_x^2} \right) + \theta_{(i,j-1)} \left( \frac{8}{h_y^2} \right)}{\frac{2}{Pe_D^2 h_x^2} + \frac{8}{h_y^2}} \quad (4.53)$$



FOR  $x^+ \geq 0$

$$x^+ \geq 0, \quad y^+ = 1, \quad \frac{\partial \theta}{\partial y^+} = -\frac{1}{4}$$

The governing equation for the given boundary condition becomes

$$\frac{1}{Pe_{D_h}^2} \frac{\partial^2 \theta}{\partial x^{+2}} + 4 \frac{\partial^2 \theta}{\partial y^{+2}} = 0 \quad (4.54)$$

$$\frac{\partial \theta}{\partial y^+} = -\frac{1}{4}, \quad \frac{\theta_{(i,j+1)} - \theta_{(i,j-1)}}{2h_y} = -\frac{1}{4}, \quad \theta_{(i,j+1)} = \theta_{(i,j-1)} - \frac{h_y}{2} \quad \text{at the boundary.}$$

After applying the finite difference approximations, the  $\theta_{(i,j)}$  at the boundary becomes

$$\theta_{(i,j)} = \frac{\theta_{(i+1,j)} \left( \frac{1}{Pe_D^2 h_x^2} \right) + \theta_{(i-1,j)} \left( \frac{1}{Pe_D^2 h_x^2} \right) + \theta_{(i,j-1)} \left( \frac{8}{h_y^2} \right) - \frac{2}{h_y}}{\frac{2}{Pe_D^2 h_x^2} + \frac{8}{h_y^2}} \quad (4.55)$$

**c) As  $x \rightarrow -\infty$**

$$\text{as } x^+ \rightarrow -\infty, \quad \theta_{(i,j)} \rightarrow 0 \quad (4.56)$$

**d) As  $x \rightarrow +\infty$**

$$\text{as } x^+ \rightarrow +\infty, \quad \theta_{(i,j)} = -2x^+ + \left[ \frac{39}{1120} + \frac{3}{16} \left( \frac{y^{+4}}{6} - y^{+2} \right) \right] \quad (4.57)$$

### 3. The Bulk Mean Temperature

The mean temperature is (2.9)

$$t_m = \frac{1}{Vh} \int_0^h ut dy.$$

When the dimensionless parameters  $u^+$ ,  $y^+$  are substituted in (2.9), the  $t_m$  becomes

$$t_m = \int_0^1 u^+ T dy^+$$

where  $V$  is the average velocity and  $h$  is the distance from the centerline to the plate.

When  $t_e = \int_0^1 t_e u^+ dy^+$  is subtracted from both sides, the equation becomes

$$(t_m - t_e) = \int_0^1 (t - t_e) u^+ dy^+$$

Let's define the bulk mean temperature,

$$\theta_m = \frac{(t_e - t_m)}{q_o'' D_h / k}, \quad (4.58a)$$

where  $t_e$  is the inlet temperature (as  $x \rightarrow -\infty$ ). Then the bulk mean temperature  $\theta_m$  becomes

$$\theta_m = \frac{3}{2} \int_0^1 (1 - y^{+2}) \theta dy^+ \quad (4.58b)$$

$\theta$  is taken from the numerical calculation. After that  $\theta_m$  is calculated by solving the equation (4.58) numerically.

#### 4. The Nusselt Number

The Nusselt number:  $Nu = \frac{h D_h}{k}$ .

The heat transfer from the wall:  $q_o'' = h(t_{(x^+, r^+=1)} - t_m)$ .

The temperature at the position of  $(x^+, r^+=1)$ :  $t_{(x^+, r^+=1)} = t_e - \theta q_o'' D_h / k$

The average temperature at ( $x^+$ ):  $t_m = t_e - \theta_m q_o'' D_h / k$

When these equations are substituted into the Nusselt number equation, Nusselt number is calculated as

$$Nu = \frac{1}{\theta_m - \theta} \quad (4.59)$$

## 5. Heat Absorption Upto the Entrance

Assume a differential volume,  $dV = b dx dy$  between  $x = -\infty$  and  $x = 0$ . The energy stored in this control volume is (4.14)

$$dE = c_p \Delta t dm$$

where  $\Delta t$  is the temperature difference between entrance temperature and the steady state temperature after the heating starts,  $\Delta t = t_{(x,y)} - t_e$  and  $dm$  is the mass of the control volume,  $dm = \rho dV$ . The energy absorbed in the control volume becomes

$$E_{absorbed} = b \rho c_p \int_{-\infty}^0 \int_{-h}^{+h} (t - t_e) b dy dx. \quad (4.60)$$

When both sides are divided by  $q_o'' D_h / k$ , the equation (4.60) becomes

$$\frac{E_{absorbed}}{q_o'' b \rho c_p h / k} = -8 \int_{-\infty}^0 \int_0^h \theta dy dx \quad (4.61)$$

where  $\theta = \frac{t_e - t}{q_o'' D_h / k}$ . As the non-dimensional parameters of  $x^+ = \frac{2x/D_h}{\text{Re Pr}}$  and  $r^+ = \frac{r}{r_o}$  and

their first derivatives are substituted into the equation (4.61), the equation becomes

$$\frac{E_{absorbed}}{q_o'' b h^3 / \alpha} = -16 Pe \int_{-\infty}^0 \int_0^1 \theta dy^+ dx^+ \quad (4.62)$$

## E. THE EFFECTS OF WALL CONDUCTION, FLOW INSIDE THE CIRCULAR TUBE, CONSTANT WALL TEMPERATURE CASE

### 1. The Governing Equation

The governing equation is (3.5)

$$\left(1 - r^{+2}\right) \frac{\partial \theta}{\partial x^+} = \frac{\partial^2 \theta}{\partial r^{+2}} + \frac{1}{r^+} \frac{\partial \theta}{\partial r^+} + \frac{1}{Pe_D^2} \frac{\partial^2 \theta}{\partial x^{+2}}$$

where

$$\theta = \frac{t_0 - t}{t_0 - t_e}, \quad r^+ = \frac{r}{r_0}, \quad x^+ = \frac{x/r_0}{Re Pr}.$$

The finite difference approximations for the first and second derivatives with error of order  $h^2$  are (4.1a,b,c,d) (figure 2.)

$$\frac{\partial \theta}{\partial x^+}_{(i,j)} = \frac{\theta_{(i+1,j)} - \theta_{(i-1,j)}}{2h_x} + o(h_x^2)$$

$$\frac{\partial \theta}{\partial r^+}_{(i,j)} = \frac{\theta_{(i,j+1)} - \theta_{(i,j-1)}}{2h_y} + o(h_y^2)$$

$$\frac{\partial^2 \theta}{\partial x^{+2}}_{(i,j)} = \frac{\theta_{(i+1,j)} + \theta_{(i-1,j)} - 2\theta_{(i,j)}}{h_x^2} + o(h_x^2)$$

$$\frac{\partial^2 \theta}{\partial r^{+2}} = \frac{\theta_{(i,j+1)} + \theta_{(i,j-1)} - 2\theta_{(i,j)}}{h_y^2} + o(h_y^2)$$

When these approximations are plugged into the governing equation (3.5), the equation becomes

$$\frac{1}{Pe_D^2} \left( \frac{\theta_{(i+1,j)} + \theta_{(i-1,j)} - 2\theta_{(i,j)}}{h_x^2} \right) + \left( \frac{\theta_{(i,j+1)} + \theta_{(i,j-1)} - 2\theta_{(i,j)}}{h_y^2} \right) + \frac{1}{y_{(i,j)}} \left( \frac{\theta_{(i,j+1)} - \theta_{(i,j-1)}}{2h_y} \right) - (1 - y_{(i,j)}^2) \left( \frac{\theta_{(i+1,j)} - \theta_{(i-1,j)}}{2h_x} \right) = 0 \quad (4.63)$$

After simplifying the expression (4.63), the equation becomes

$$\theta_{(i,j)} = \frac{\theta_{(i+1,j)} \left( \frac{1}{Pe_D^2 h_x^2} + \frac{y_{(i,j)}^2}{2h_x} - \frac{1}{2h_x} \right) + \theta_{(i-1,j)} \left( \frac{1}{Pe_D^2 h_x^2} - \frac{y_{(i,j)}^2}{2h_x} + \frac{1}{2h_x} \right) + \theta_{(i,j+1)} \left( \frac{1}{h_y^2} + \frac{1}{2h_y y_{(i,j)}} \right) + \theta_{(i,j-1)} \left( \frac{1}{h_y^2} - \frac{1}{2h_y y_{(i,j)}} \right)}{\frac{2}{Pe_D^2 h_x^2} + \frac{2}{h_y^2}} \quad (4.64)$$

## 2. The Boundary Conditions

The boundary conditions are:

### a) At the Center

$$\text{At } r^+ = 0, \quad \frac{\partial \theta}{\partial r^+} = 0 \quad (\text{Symmetry boundary condition})$$

When this boundary condition is applied to the governing equation, the element  $\frac{1}{r^+} \frac{\partial \theta}{\partial r^+}$  becomes  $\frac{0}{0}$ . After applying the L'Hospital's rule, the governing equation for the given boundary condition becomes

$$\frac{\partial \theta}{\partial x^+} = 2 \frac{\partial^2 \theta}{\partial r^{+2}} + \frac{1}{Pe^2} \frac{\partial^2 \theta}{\partial x^{+2}} \quad (4.65)$$

Since the temperature distribution at the center on the radial direction is symmetric,

$$\theta_{(i,j+1)} = \theta_{(i,j-1)} .$$

After applying the finite difference method, the  $\theta_{(i,j)}$  at the boundary becomes

$$\theta_{(i,j)} = \frac{\theta_{(i+1,j)} \left( \frac{1}{Pe_D^2 h_x^2} - \frac{1}{2h_x} \right) + \theta_{(i-1,j)} \left( \frac{1}{Pe_D^2 h_x^2} + \frac{1}{2h_x} \right) + \theta_{(i,j+1)} \left( \frac{4}{h_y^2} \right)}{\frac{2}{Pe_D^2 h_x^2} + \frac{4}{h_y^2}} \quad (4.66)$$

### b) At the Wall

$$\text{at } r^+ = 1, \quad \begin{cases} \theta_{wall} = \theta_{fluid} \\ \frac{\partial \theta_{wall}}{\partial r^+} = \frac{\partial \theta_{fluid}}{\partial r^+} \end{cases}$$

The heat fluxes through the fluid and the wall are equal to each other at  $r^+ = 1$ .

$$k_{solid} \frac{\partial \theta_{solid}}{\partial r^+} = k_{fluid} \frac{\partial \theta_{fluid}}{\partial r^+}$$

Using the 2<sup>nd</sup> order forward and backward difference method, the equality becomes

$$k_f \left( \frac{3\theta_{(i,j)} - 4\theta_{(i,j-1)} + \theta_{(i,j-2)}}{2h_{yf}} \right) = k_s \left( \frac{-3\theta_{(i,j)} + 4\theta_{(i,j+1)} - \theta_{(i,j+2)}}{2h_{ys}} \right) \quad (4.67)$$

If  $h_{ys} = h_{yf}$ , the bulk temperature at the fluid-wall boundary becomes

$$\theta_{(i,j)} = \frac{4 \left( \theta_{(i,j+1)} + \frac{k_f}{k_s} \theta_{(i,j-1)} \right) - \left( \theta_{(i,j+2)} + \frac{k_f}{k_s} \theta_{(i,j-2)} \right)}{3 \left( 1 + \frac{k_f}{k_s} \right)} \quad (4.68)$$

**c) On the Outer Edge of the Wall**

FOR  $x^+ < 0$

For  $x^+ < 0$  and  $r^+ = 1 + \delta y$ , the outer surface of the solid wall is insulated where

$$\delta y = \frac{\text{solid\_wall\_thickness}}{\text{tube\_inner\_radius}}.$$

The governing equation for the given boundary condition becomes

$$\frac{\partial^2 \theta}{\partial r^{+2}} + \frac{1}{Pe^2} \frac{\partial^2 \theta}{\partial x^{+2}} = 0 \quad (4.69)$$

Since the wall is insulated,  $\frac{\partial \theta}{\partial r^+} = 0$ ,  $\theta_{(i,j+1)} = \theta_{(i,j-1)}$  at the boundary.

After applying the finite difference approximations, the  $\theta_{(i,j)}$  at the boundary becomes

$$\theta_{(i,j)} = \frac{\theta_{(i+1,j)} \left( \frac{1}{Pe_D^2 h_x^2} \right) + \theta_{(i-1,j)} \left( \frac{1}{Pe_D^2 h_x^2} \right) + \theta_{(i,j-1)} \left( \frac{2}{h_y^2} \right)}{\frac{2}{Pe_D^2 h_x^2} + \frac{2}{h_y^2}} \quad (4.70)$$

FOR  $x^+ > 1$

$$\theta_{(i,j)} = 0 \quad (4.71)$$

**d) As  $x \rightarrow -\infty$**

$$\text{as } x^+ \rightarrow -\infty, \quad \theta_{(i,j)} \rightarrow 1 \quad (4.72)$$

**e) As  $x \rightarrow +\infty$**



$$\text{as } x^+ \rightarrow +\infty, \quad \theta_{(i,j)} \rightarrow 0 \quad (4.73)$$

### 3. The Bulk Mean Temperature

The mean temperature at a position “i” is (2.8)

$$t_{m(x)} = \frac{2}{Vr_0^2} \int_0^{r_0} u_{(x,r)} t_{(x,r)} r dr$$

where  $V$  is the average velocity and  $r_0$  is the radius of the tube. When  $\int_0^{r_0} u_{(x,r)} 2\pi r dr$  is subtracted from both sides and dimensionless parameter  $r^+$  is substituted inside the equation, the equation becomes

$$(t_m - t_o) = \int_0^1 (t - t_o) u^+ 2r^+ dr^+$$

Let's define the bulk mean temperature,  $\theta_m = \frac{(t_m - t_o)}{(t_e - t_o)}$ , where  $t_e$  is the inlet temperature

(as  $x \rightarrow -\infty$ ) and  $t_o$  is the constant wall temperature. (Fluid exit temperature as  $x \rightarrow +\infty$ )

Then the bulk mean temperature  $\theta_m$  becomes

$$\theta_{m(i)} = 4 \int_0^1 (1 - r^{+2}) \theta r^+ dr^+ \quad (4.74)$$

$\theta$  is found out from the numerical calculation. After that  $\theta_m$  is calculated by solving the equation (4.74) numerically.

#### 4. The Nusselt Number

The Nusselt number is defined as (2.3)

$$Nu = \frac{hd}{k}.$$

When the heat transfer equations  $q_o'' = h(t_o - t_m)$  and  $q_o'' = -k \frac{\partial \theta}{\partial r} \Big|_{r=R}$ , and the

dimensionless parameter  $r^+$  are plugged in (2.3), the equation becomes

$$Nu = - \frac{2}{\theta_m} \frac{\partial \theta}{\partial r^+} \Big|_{r^+=1}. \quad (4.75)$$

The finite difference approximation for the equation (4.75) using the backward difference expression with error of order  $h^2$  becomes

$$Nu_{(i)} = \frac{2}{\theta_{m(i)}} \left( \frac{-3\theta_{(i,j)} + 4\theta_{(i,j-1)} - \theta_{(i,j-2)}}{2h_y} \right) \quad (4.76)$$

where “j” is the value which corresponds to  $r^+=1$ .

#### 5. Heat Absorption Up to the Entrance

Assume a differential volume,  $dV = 2\pi r dr$  between  $x = -\infty$  and  $x = 0$ . The energy stored in this control volume is (4.14)

$$dE = c_p \Delta t dm$$

where  $\Delta t$  is the temperature difference between entrance temperature and the steady state temperature after the heating starts,  $\Delta t = t_{(x,r)} - t_e$  and  $dm$  is the mass of the control volume,  $dm = \rho dV$ . The energy absorbed in the control volume becomes

$$E_{absorbed} = 2\pi\rho c_p \int_{-\infty}^0 \int_0^{r_o} (t - t_e) r dr dx . \quad (4.77)$$

When  $2\pi\rho c_p \int_{-\infty}^0 \int_0^{r_o} t_o r dr dx$  is subtracted from both sides and  $(t_e - t_o)$  is divided to both sides, the equation (4.77) becomes

$$\frac{E_{absorbed}}{(t_e - t_o)\rho c_p} = 2\pi \int_{-\infty}^0 \int_0^{r_o} (\theta - 1) r dr dx \quad (4.78)$$

where  $\theta = \frac{t - t_o}{t_e - t_o}$ . As the non-dimensional parameters of  $x^+ = \frac{x/r_o}{\text{RePr}}$  and  $r^+ = \frac{r}{r_o}$  and

their first derivatives are substituted into the equation (4.16), the equation becomes

$$\frac{E_{absorbed}}{(t_o - t_e)\rho c_p r_o^3} = -2\pi Pe \int_{-\infty}^0 \int_0^1 (\theta - 1) r^+ dr^+ dx^+ \quad (4.79)$$



## V. NUMERICAL RESULTS AND DISCUSSION

### A. BULK MEAN TEMPERATURE VS. X

The bulk mean temperature graphs of the fully developed laminar flow inside the circular tube and parallel plates for constant wall temperature and heat flux cases were plotted in figures 8-11. The Peclet numbers 0.5, 1, 2, 3, 5, 10, 50, and 100 were used for the plots to see how the Peclet number,  $Pe=RePr$ , affects the axial conduction in the flow. As it is seen from the figures, when the Peclet number gets smaller, the axial conduction effects of the fluid becomes very important and the temperature of the fluid upto the entrance increases. The axial conduction of the fluid has to be considered for these cases and must not be neglected for the Peclet numbers less than about 10. When the Peclet number gets larger, the effects of the axial conduction decrease. As it is seen from the figures 8-11, for the Peclet numbers 50 and 100, the bulk mean temperature of the fluid at the position  $x^+=0$  is almost equal to the entrance temperature, which means that there is almost no heat absorption or temperature increase upto the entrance. The axial conduction of the fluids can be neglected for these high Peclet numbers.

#### 1. Flow Inside the Circular Tube, Constant Wall Temperature

##### Case

The graphs of the bulk mean temperature vs.  $x$  for the circular tube, constant wall temperature flow was plotted for various Peclet numbers in figure 8.a and 8.b where theta

bulk mean is  $\theta_m = \frac{(t_m - t_o)}{(t_e - t_o)}$  and the nondimensional axial distance  $x^+$  is  $x^+ = \frac{2x/d}{\text{Re Pr}}$ .

The validity of the figure 8.b has been checked by comparison with Ref.5.

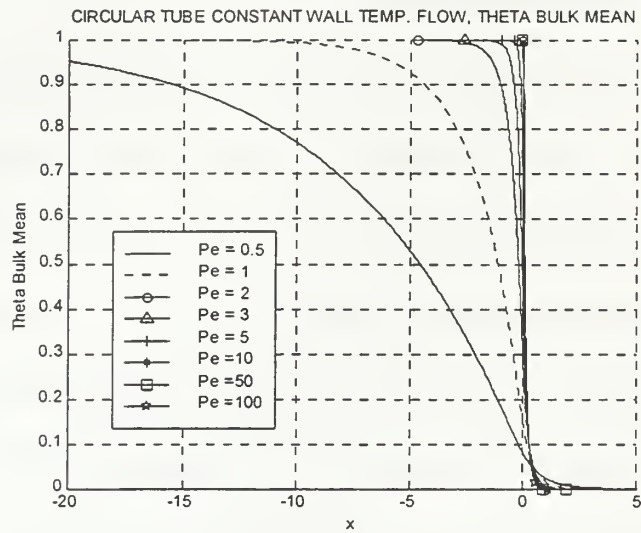


Figure 8.a The axial bulk mean temperature distribution for circular tube constant wall temperature flow

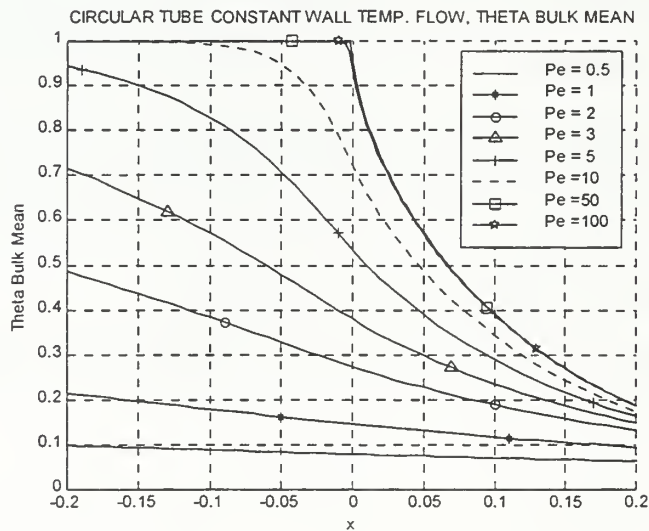


Figure 8.b The enlarged view of figure 8.a

## 2. Flow Inside the Circular Tube, Constant Heat Flux Case

The graphs of the bulk mean temperature vs.  $x$  for the circular tube, constant heat flux flow was plotted for various Peclet numbers in figure 9.a and 9.b where theta bulk mean is  $\theta_m = \frac{(t_e - t_m)}{q_o'' d/k}$  and the nondimensional axial distance  $x^+$  is  $x^+ = \frac{2x/d}{Re_d Pr}$ .

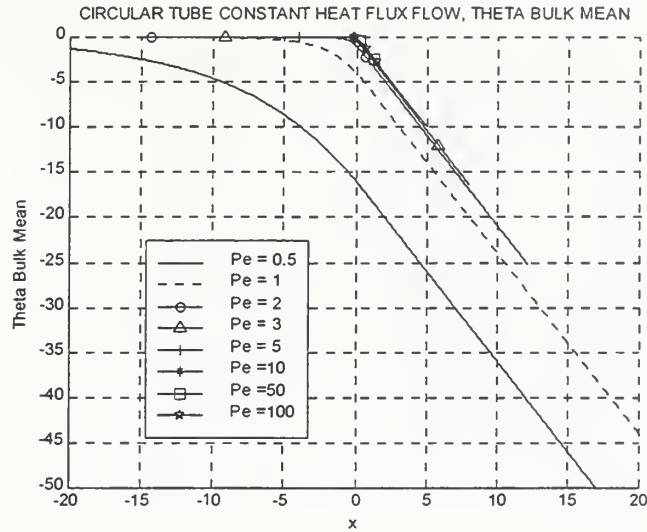


Figure 9.a The axial bulk mean temperature distribution for circular tube constant heat flux flow

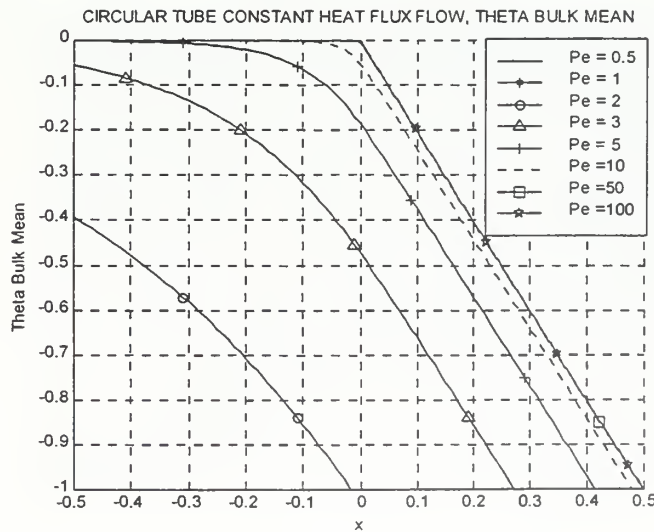


Figure 9.b The enlarged view of figure 9.a

### 3. Flow Between the Parallel Plates, Constant Wall Temperature

#### Case

The graphs of the bulk mean temperature vs.  $x^+$  for the parallel plates, constant wall temperature flow was plotted for various Peclet numbers in figure 10.a and 10.b where

theta bulk mean is  $\theta_m = \frac{(t_o - t_m)}{(t_o - t_c)}$  and the nondimensional axial distance  $x^+$  is  $x^+ = \frac{2x / D_h}{Re Pr}$ .

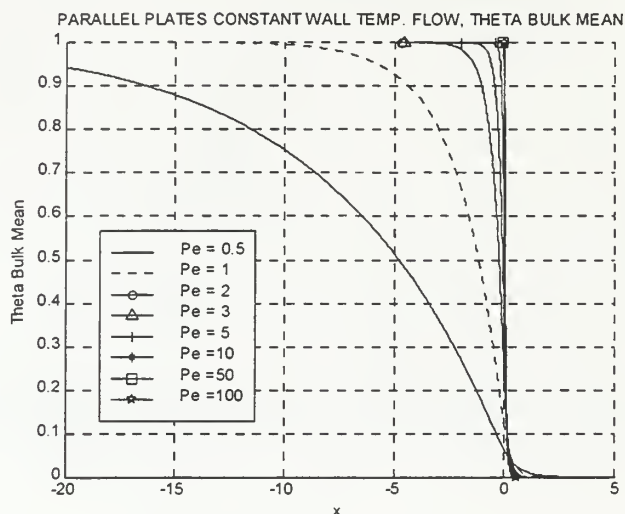


Figure 10.a The axial bulk mean temperature distribution for parallel plates constant wall temperature flow

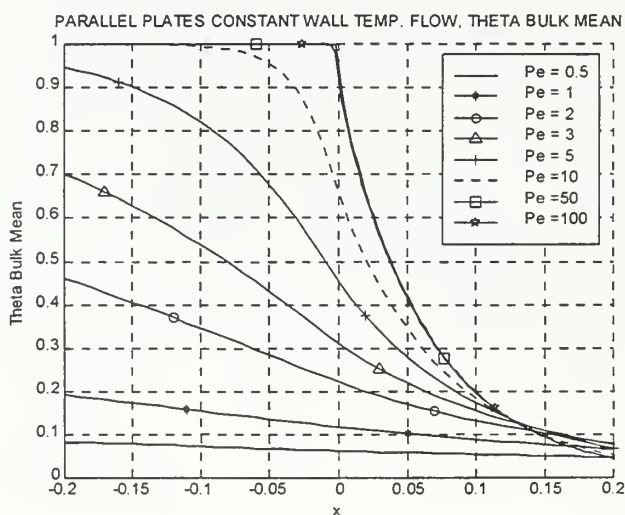


Figure 10.b The enlarged view of figure 10.a



#### 4. Flow Between the Parallel Plates, Constant Heat Flux Case

The graphs of the bulk mean temperature vs.  $x$  for the parallel plates, constant heat flux flow was plotted for various Peclet numbers in figure 11.a and 11.b where theta bulk mean is  $\theta_m = \frac{(t_e - t_m)}{q_o'' D_h / k}$  and the nondimensional axial distance  $x^+$  is  $x^+ = \frac{2x / D_h}{\text{RePr}}$ .

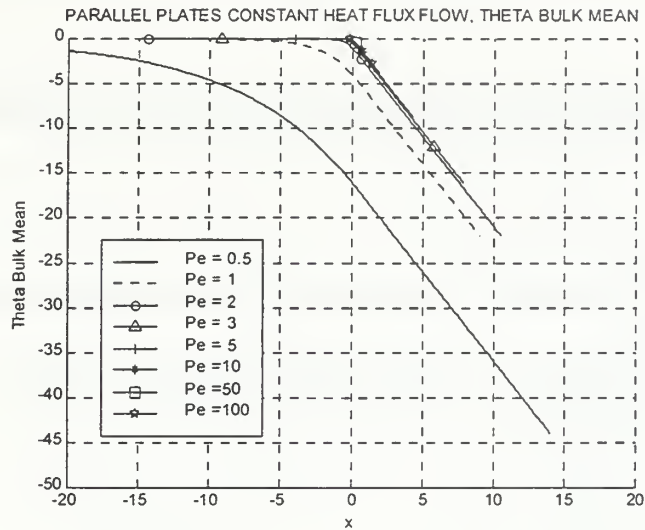


Figure 11.a The axial bulk mean temperature distribution for parallel plates constant heat flux flow

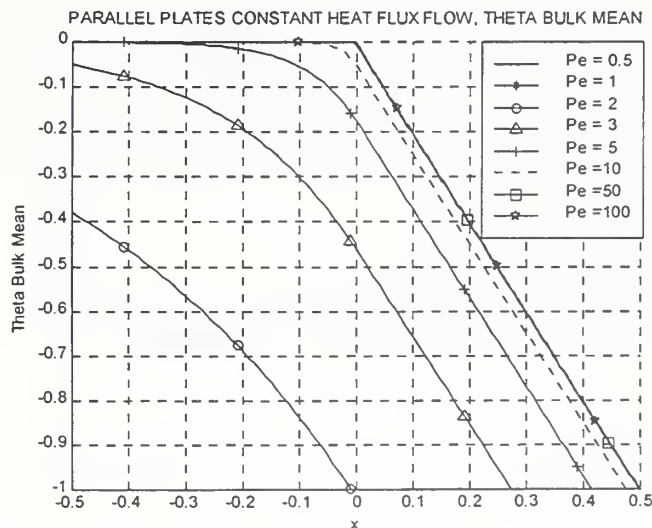


Figure 11.b The enlarged view of figure 11.a

## B. NUSSELT NUMBER VS. X

The nusselt number graph of the fully developed laminar flow inside the circular tube and parallel plates for constant wall temperature and heat flux cases were plotted in figures 12-15. The Peclet numbers 0.5, 1, 2, 3, 5, 10, 50, and 100 were used for the plots to see how the Peclet number,  $Pe=RePr$ , affects the nusselt number distribution in the flow.

### 1. Flow Inside the Circular Tube, Constant Wall Temperature

#### Case

The graph of the nusselt number vs.  $x$  for the circular tube, constant wall temperature flow was plotted for various Peclet numbers in figure 12 where  $x^+$  is

$$x^+ = \frac{2x/d}{RePr}.$$

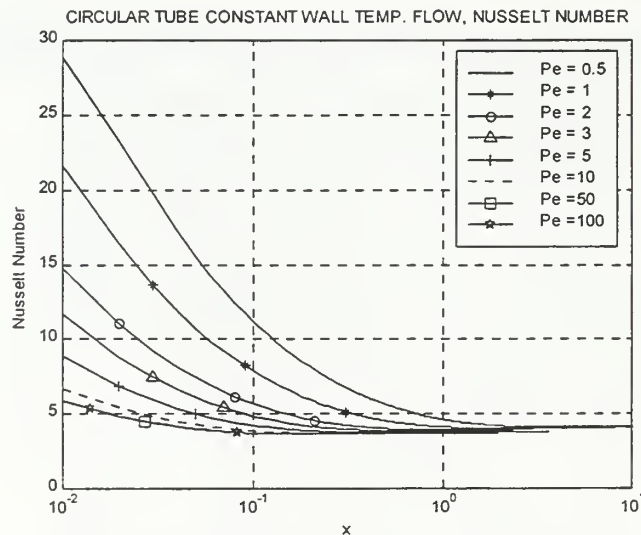


Figure 12. The nusselt number distribution for circular tube constant wall temperature flow

The nusselt numbers approach to the values in those of table 1 when  $x$  gets larger as seen in figure 12.

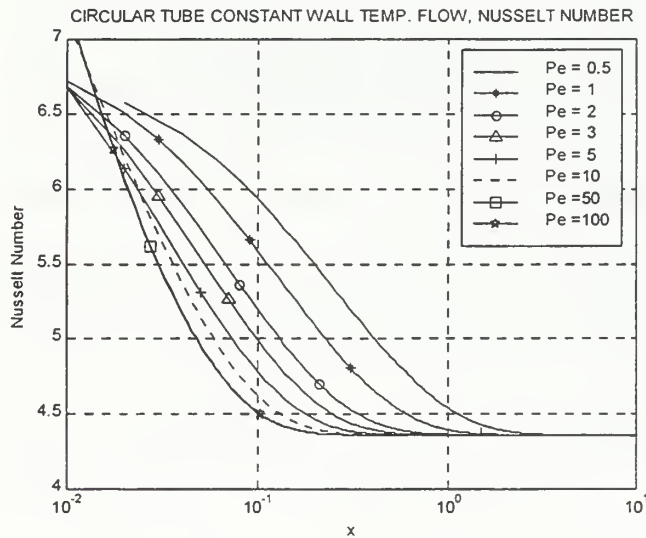
$Pe$	0.5	1	2	3	5	10	50	100
Nu. no	4.0971	4.0276	3.9224	3.8506	3.767	3.6948	3.6586	3.6572

**Table 1.** The nusselt number for various Peclet numbers for circular tube constant wall temperature case as  $x$  goes to infinity

## 2. Flow Inside the Circular Tube, Constant Heat Flux Case

The graph of the nusselt number vs.  $x$  for the circular tube, constant heat flux flow

was plotted for various Peclet numbers in figure 13 where  $x^+$  is  $x^+ = \frac{2x/d}{Re_d Pr}$ .



**Figure 13.** The nusselt number distribution for circular tube constant heat flux flow

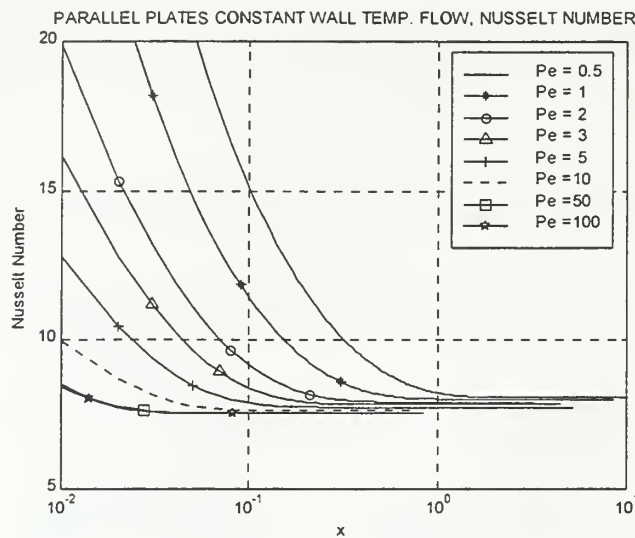
The nusselt number is independent from the Peclet number for this case, and approaches to 4.364 where  $x$  goes to infinity.

### 3. Flow Between the Parallel Plates, Constant Wall Temperature

#### Case

The graph of the nusselt number vs.  $x$  for the parallel plates, constant wall temperature flow was plotted for various Peclet numbers in figure 14 where  $x^+$  is

$$x^+ = \frac{2x/D_h}{\text{Re Pr}}, D_h \text{ is } 4h.$$



**Figure 14.** The nusselt number distribution for parallel plates constant wall temperature flow

The nusselt numbers approach to the values in those of table 2 when  $x$  gets larger as seen in figure 12.

$Pe$	0.5	1	2	3	5	10	50	100
Nu. no	8.0588	8.0059	7.9168	7.8464	7.7471	7.6303	7.5457	7.542

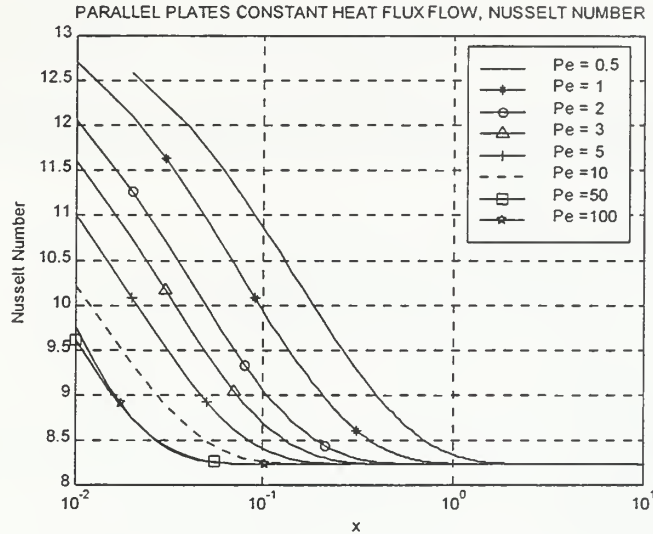
**Table 2.** The nusselt number for various Peclet numbers for parallel plates constant wall temperature case as  $x$  goes to infinity

### 3. Flow Between the Parallel Plates, Constant Heat Flux Case

The graph of the nusselt number vs.  $x$  for the parallel plates, constant heat flux

flow was plotted for various Peclet numbers in figure 15 where  $x^+$  is  $x^+ = \frac{2x / D_h}{Re Pr}$ ,  $D_h$  is

4h.



**Figure 15.** The nusselt number distribution for parallel plates constant heat flux flow

The nusselt number is independent from the Peclet number for this case, and approaches to 8.235 where  $x$  goes to infinity.

### C. HEAT ABSORPTION UPTO THE ENTRANCE

The energy change from the entrance to  $x=0$  where the heating starts has been investigated and plotted for fully developed laminar flow inside the circular tube and parallel plates for constant wall temperature and heat flux cases in figures 16-19. The Peclet numbers 0.5, 1, 2, 3, 5, 10, 50, and 100 were used for the plots to see how the Peclet number,  $Pe=RePr$ , affects the heat absorption of the fluid. As it is seen from the graphs, the heat absorption

logarithmically increases as the Peclet number decreases. Therefore axial conduction effects or heat absorption upto the entrance where the heating starts has to be considered for low Peclet numbers.

## 1. Flow Inside the Circular Tube, Constant Wall Temperature

### Case

The graph of the heat absorption,  $E^*$  vs. Peclet number for the circular tube, constant wall temperature flow was plotted in figure 16 where the dimensionless absorbed energy  $E^*$  is

$$E^* = \frac{E_{absorbed}}{(t_o - t_c) \rho c_p r_o^3}$$

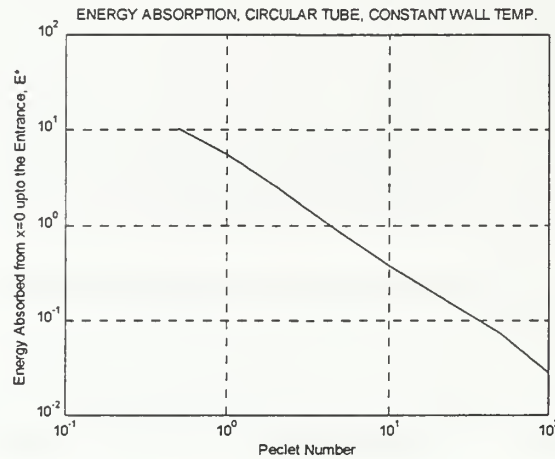
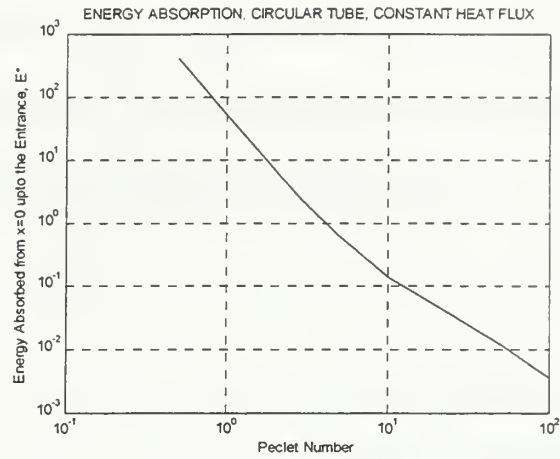


Figure 16. The energy absorption upto the entrance for circular tube constant wall temperature flow

## 2. Flow Inside the Circular Tube, Constant Heat Flux Case

The graph of the heat absorption,  $E^*$  vs. Peclet number for the circular tube, constant heat flux flow was plotted in figure 17 where the dimensionless absorbed energy  $E^*$  is

$$E^* = \frac{E_{absorbed}}{q_o'' \rho c_p r_o^4 / k}$$



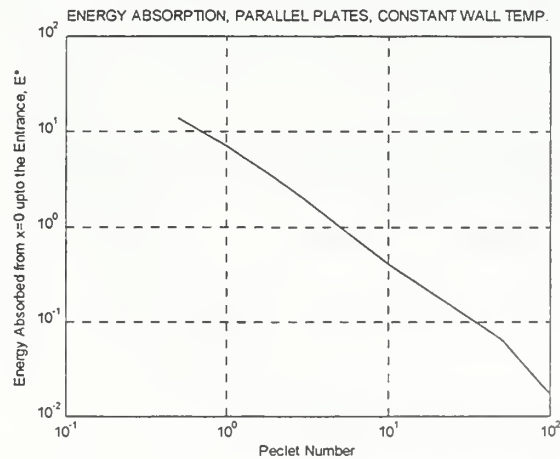
**Figure 17.** The energy absorption upto the entrance for circular tube constant heat flux flow

### 3. Flow Between the Parallel Plates, Constant Wall Temperature

#### Case

The graph of the heat absorption,  $E^*$  vs. Peclet number for the parallel plates, constant wall temperature flow was plotted in figure 18 where the dimensionless absorbed energy  $E^*$  is

$$E^* = \frac{E_{\text{absorbed}}}{h^2 b \rho c_p (t_o - t_e)}.$$



**Figure 18.** The energy absorption upto the entrance for parallel plates constant wall temperature flow

#### 4. Flow Between the Parallel Plates, Constant Heat Flux Case

The graph of the heat absorption,  $E^*$  vs. Peclet number for the parallel plates, constant heat flux flow was plotted in figure 19 where the dimensionless absorbed energy  $E^*$  is

$$E^* = \frac{E_{\text{absorbed}}}{q_o'' b h^3 / \alpha}.$$

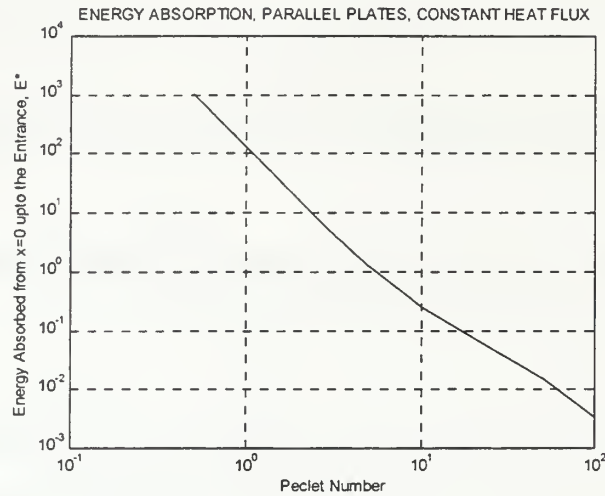


Figure 19. The energy absorption upto the entrance for parallel plates constant heat flux flow

#### D. AXIAL CONDUCTION EFFECTS INCLUDING THE WALL CONDUCTION

The axial conduction effects for circular tube constant wall temperature case has been investigated including the wall conduction effects. The Peclet numbers 1, 3, 5, outer/inner radius ratios of 1.2, 2 and fluid/wall thermal conductivity ratios of 1 and 1/10 were used to see how these properties affect the temperature, nusselt number and heat absorption. The theta bulk mean temperature, nusselt number variations were plotted vs.



$x^+$  and the heat absorption were quantified as a function of Peclet number, thermal conductivity and radius ratios. The results are seen on the following pages.

## 1. Bulk Mean Temperature Variations

The  $\frac{\theta_{cl}}{\theta_{cl,x=0}}$  vs.  $x^+$  graphs are seen in figures 20.a and 20.b for different Peclet

numbers, thermal conductivity and wall thickness. Since the bulk mean temperature values are very small about  $x^+=0$ , the normalized bulk mean temperature on the centerline is a good way to see the change at the bulk temperature relative to the centerline

temperature at  $x^+=0$ , where the bulk temperature is  $\theta = \frac{(t-t_o)}{(t_e-t_o)}$  and  $x^+$  is  $x^+ = \frac{2x/d}{Re Pr}$ .

The bulk mean temperature vs.  $x^+$  graphs for the Peclet numbers 1,3, and 5, and the thermal conductivity ratios of 1, 1/10 and the radius ratios of 1.2 and 2 are seen in figure

21 where the bulk mean temperature is  $\theta_m = \frac{(t_m-t_o)}{(t_e-t_o)}$ .

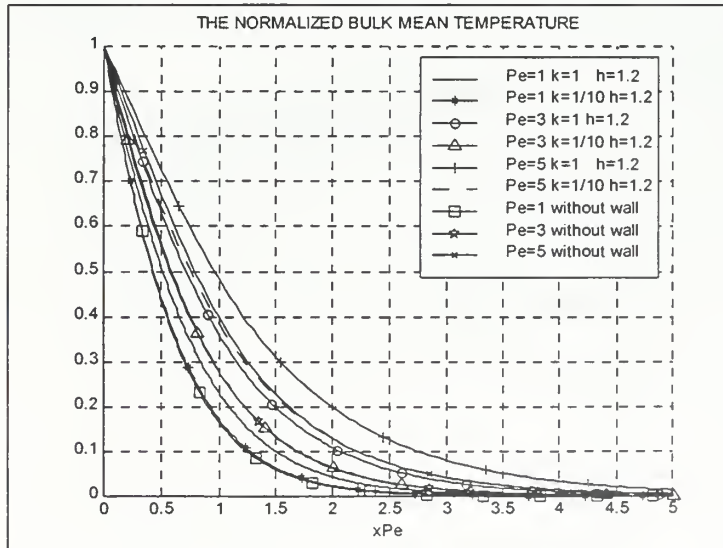


Figure 20.a The normalized bulk mean temperature with and without wall

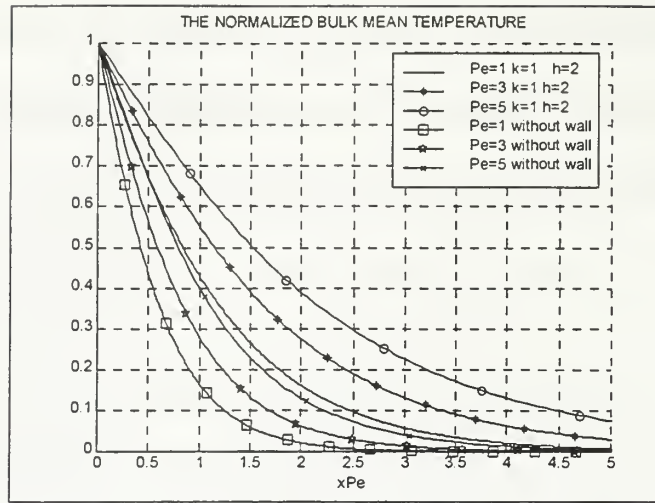


Figure 20.b The normalized energy absorption upto the entrance for parallel plates constant heat flux flow

The theta bulk mean values with the wall conduction case is closer to the steady state values than the no wall conduction case as it is seen in figure 21. The bulk mean temperature go to zero as  $x$  goes to plus infinity. This means that the temperature of the fluid including the wall conduction increases faster than the temperature of the fluid neglecting the wall conduction. The main reason of the relative increase at the temperature including the wall conduction case is the heat that is transferred by the axial conduction of the wall opposite to the flow direction. The heat moves to the opposite of the flow direction and is absorbed by the fluid. So the fluid temperature increases much more than no wall case.

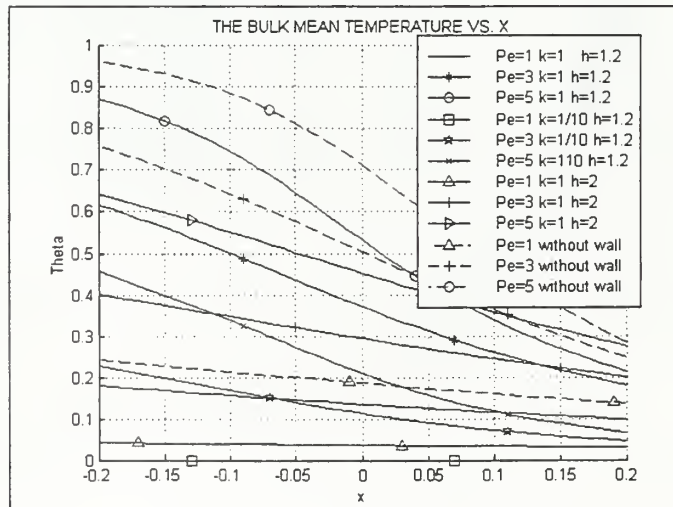


Figure 21. The bulk mean temperature with and without wall conduction

## 2. Nusselt Number vs. X

The Nusselt numbers vs.  $x^+$  graphs are seen in figure 22 for different Peclet numbers,

thermal conductivity and wall thickness where  $x^+$  is  $x^+ = \frac{2x/d}{Re Pr}$ . The Nusselt number

distribution is very different from the no wall conduction case as it is seen in the figure.

As the wall gets thinner and the thermal conductivity increases, the values of the Nusselt

number gets closer to the case of no wall case.  $h=1/10$  and  $k=1.2$  is close to the no wall

conduction case, so the values of the Nusselt number for these curves are close to the

values of no wall case.

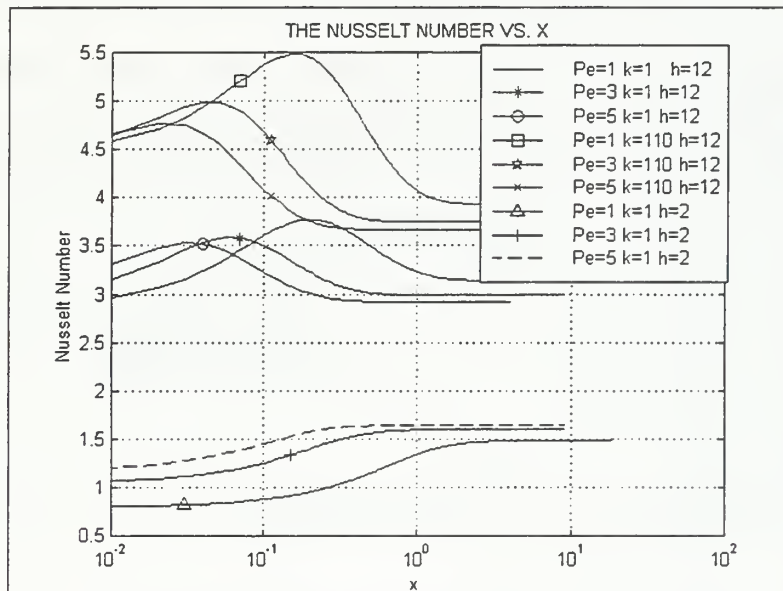


Figure 22. The nusselt number distribution including the wall conduction

### 3. Heat Absorption vs. X

The graphs of the heat absorption,  $E^*$  vs. Peclet numbers for the conductivity ratios of 1, 1/10 and radius ratios of 1.2 and 2 were plotted in figure 23 where the dimensionless absorbed energy  $E^*$  is  $E^* = \frac{E_{absorbed}}{(t_o - t_c) \rho c_p r_o^3}$ . As it is seen from the graph, As the wall's conductivity or thickness increases, the heat absorption before  $x=0$  significantly increases. The wall conduction case has to be added to the problem for the microheat exchanger problems. Because the relative thickness of the wall is very large as it is compared to the inner diameter of the tube. The values of  $h=1.2$  or  $h=2$  are realistic for the microheat machines, because the diameter of the tubes are very small for these heat exchanger. The heat absorption for  $k_f/k_s=1/10$  and  $h=1.2$  case is more than ten times of the case of no wall. If the wall thickness would be  $h=2$ , then the heat absorption would be much greater.

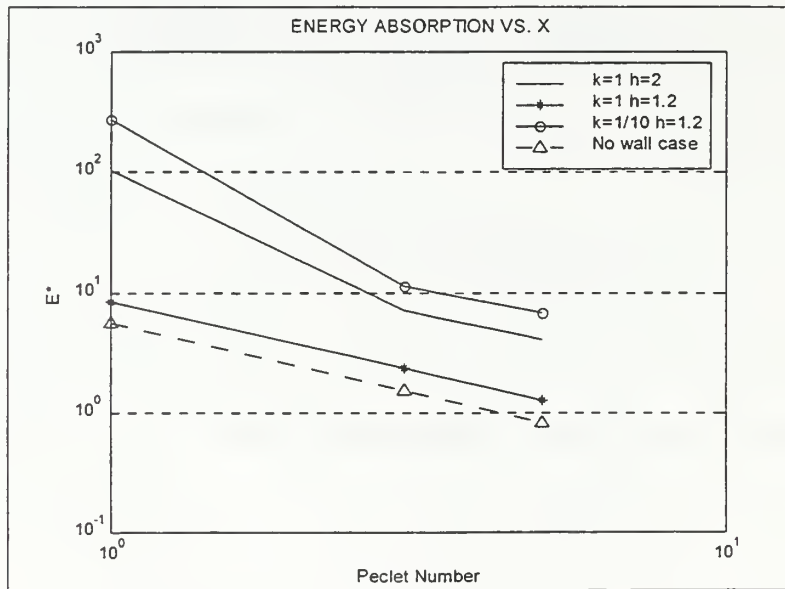


Figure 23. The energy absorption upto the entrance with and without the axial wall conduction

## VI. CONCLUSIONS

Axial conduction effects in duct flows are important and have to be considered for cases of  $Pe < 10$ . The theta bulk mean temperature and Nusselt number variations vs. downstream distance were plotted as seen in the results. It is obviously seen the increase in the temperature against the flow direction from  $x=0$  as the Peclet number decreases. Heat absorption by the fluid in the insulated region was also quantified by using the non-dimensional parameter  $E^*$ . The heat absorption increases logarithmically as the Peclet number decreases, as seen in the heat absorption graphs. When the wall conduction effects are included in the problem, it was calculated that most of the heat absorption by the fluid is because of the wall conductivity as the wall conductivity gets larger. The same effect is also seen with increasing wall thickness. The heat absorption by the fluid gets larger when the wall gets thicker. It is very important to include the wall conduction in the problem as the wall gets thicker and the thermal conductivity of the wall gets larger. With large wall conductivity and wall thickness, the fluid temperature increases significantly before the heating starts. The current interests in axial conduction effects are because of the advent of micro heat exchanger technology with microtubes where the Reynolds number is very small.



## APPENDIX – A SAMPLE PROGRAM

### The Program for Circular Tube, Constant Wall Temperature Case

This program solves the finite difference equations for the circular tube, constant wall temperature case by using the Gauss Seidel Iteration Method with the overrelaxation parameter  $w=1.8$ . The program solves the equations doing the following steps :

1. The matrix is formed by the program, the first row represents the region which touches the tube. The last row is the centerline. The first column is the condition, where  $x$  goes to minus infinity. The last column represents the boundary condition as  $x$  goes to plus infinity.

2. The boundary conditions are applied to the problem. The centerline boundary condition, the boundary conditions at  $r=r_o$ , the boundary conditions as  $x$  goes to minus and plus infinity are applied to the last row, first row, first column and the last column respectively.

3. The iterations are made calculating the values at all the grid points. As it is seen in the program, the odd rows are calculated first, the even rows later. The convergence of the problem is better in this case.

4. The difference between the error and the boundary conditions for the plus and minus infinity is checked. If the boundary conditions and the difference between the iterations are reasonable, the program is stopped and the data is processed. The max. difference between the iterations is less than  $1E-6$ . The max difference between the values of the horizontal grid points which represents minus and plus infinity is less than  $1E-6$ .



```
% This problem solves the problem for circular tube
% constant wall temperature case
%
% Programmed by Ibrahim Girgin
%

format long

clear
start=1;
iteration_number=300000;

% Nx is the column number of the matrix
% Ny is the row number of the matrix
% x_zero is the column number which represents x=0

Nx=2501
Ny=101
x_zero=1501

% For Peclet number is 1

Pe=1

% For the relaxation parameter 1.8

w=1.8
save_iteration=100

% hy is the vertical distance between the grids
% hx is the horizontal distance between the grids

hy=1/(Ny-1);
hx=hy;

% teta is the matrix which holds the grid points
% initialization of the teta

teta=ones(Ny,Nx);

% Apply the boundary condition for r=R0, x>=0

teta(:,x_zero:Nx)=zeros(Ny,length(x_zero:Nx));


% e.g.          j=Ny > 1   1   1   1   1   1   1   1   0   0   0   0   0   0   0   0
% the initial      1   1   1   1   1   1   1   1   0   0   0   0   0   0   0   0
% value of         1   1   1   1   1   1   1   1   0   0   0   0   0   0   0   0
% theta           1   1   1   1   1   1   1   1   0   0   0   0   0   0   0   0
%                 1   1   1   1   1   1   1   1   0   0   0   0   0   0   0   0
%                j=1 > 1   1   1   1   1   1   1   1   0   0   0   0   0   0   0   0
%                  ^               ^               ^
%                   i=1             i=x_zero         i=Nx

% The values for this theta matrix is :
```



```

%      Nx=14      Ny=6      x_zero=8      hx=0.2      hy=0.2
%      Pe,w,file_name,save iteration is what is to be given

% The variables that are being used inside the loop

var1=Pe^2*hx^2;
var2=2/hy^2;
var3=1/2/hx;
var4=1/hy^2;
var5=1/var1-var3;
var6=1/var1+var3;
var7=1/2/hy;
var8=2/var1+4*var4;
var9=2/var1+var2;
var10=2*hx;

% The iteration starts here. For every value of kkk, one
% iteration is made

for kkk=start:iteration_number

    for _max_err=teta;

        for i=2:Nx-1      % do the iteration for all the columns except the
                           % and the last columns

            for j=1:2:Ny      % do the iteration for all the odd rows

                y=(j-1)*hy; % calculate the y,      0<= y <=1

                if j==1      % do the iteration for radius=0 (for the center)

                    teta(Ny,i)=(teta(Ny,i+1)*(var5)+teta(Ny,i-1)*(var6)+ ...
                    4*teta(Ny-1,i)*var4)/var8;

                elseif j==Ny      % do the iteration for radius=R

                    if i<x_zero % if i<x_zero & for radius R, it is insulated

                        teta(1,i)= ((teta(1,i+1)+teta(1,i-
                        1))/(var1)+teta(2,i)*var2)/var9;

                    end

                else      % do the iteration for all the odd rows except r=0
and r=R

                    teta(Ny-j+1,i)=teta(Ny-j+1,i)+w*((teta(Ny-j+1,i+1)*(var5 +
                    y^2/var10) ...

```

```

        +teta(Ny-j+1,i-1)*(var6 - y^2/var10)+teta(Ny-j,i)*(var4 +
var7/y)+...
        teta(Ny-j+2,i)*(var4 - var7/y))/var9-teta(Ny-j+1,i));

end

end

for j=2:2:(Ny-1)      %    do the iteration for all the even rows

    y=(j-1)*hy;
    teta(Ny-j+1,i)=teta(Ny-j+1,i)+w*((teta(Ny-j+1,i+1)*(var5 +
y^2/var10)...
    +teta(Ny-j+1,i-1)*(var6 - y^2/var10)+teta(Ny-j,i)*(var4 + var7/y)+
...
    teta(Ny-j+2,i)*(var4 - var7/y))/var9-teta(Ny-j+1,i));

end

end      %    End of the iteration

%    Save the variables at every "save_iteration"
%    e.g. if the save_iteration equals to 20, then save
%    the variables at when kkk=20,40,60,80,... so on.

if fix(kkk/save_iteration)==kkk/save_iteration;

    max_error=(max(max(abs(teta-for_max_err))))
    clear for_max_err
    save forpel teta Nx Ny x_zero w Pe hx hy max_error kkk
save_iteration start y

end

end

```

## LIST OF REFERENCES

- [1] Frank P. Incropera, David P. Dewitt, *Introduction to Heat Transfer*, John Wiley & Sons, 1996, Canada.
- [2] W. M. Kays, M. E. Crawford, *Convective Heat and Mass Transfer*, McGraw-Hill, Inc. 1993.
- [3] Adrian Bejan, *Convection Heat Transfer*, John Wiley & Sons, 1996, Canada.
- [4] L. Graetz, *Über die Wärmeleitungsfähigkeit von Flüssigkeiten* (On the thermal Conductivity of liquids) Part 1. Ann. Phys. Chem., Vol. 18, pp.79-94, 1883, Part 2; Ann. Phys. Chem., Vol. 25, pp. 337-357, 1885.
- [5] M. L. Michelsen and John Villadsen, "The Graetz Problem With Axial Heat Conduction," *International Journal of Heat Mass Transfer*, Vol. 17, pp. 1391-1402, 1974.
- [6] X. Yin and H. H. Bau, "The Conjugate Graetz Problem With Axial Conduction," *Journal of Heat Transfer*, Vol. 118, pp. 482-485, May 1996.
- [7] M. L. James, G. M. Smith, J. C. Welford, *Applied Numerical Methods for Digital Computation*, Fourth Edition, HarperCollins College Publishers, 1993.
- [8] George E. Forsythe and Wolfgang R. Wasov, *Finite Difference Methods for Partial Differential Equations*, Wiley, pp. 256, 1960, New York.



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